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Stochastic and Integral Geometry I (WS 06/07)

Exercise Sheet 7

Please, hand in your solutions at the end of the **lecture** on **Monday, Dec 18th**!

1. Prove the following generalization of the Steiner formula: Let $K \in \mathcal{K}$ and $k \in \{0, 1, \dots, d-1\}$. Then, for each $A \in \mathcal{B}(\mathbb{R}^d)$ with $A \subset K$ and $\varepsilon \geq 0$,
$$\Phi_k(K + \varepsilon B^d, A + \varepsilon S^{d-1}) = \sum_{j=0}^{k} \varepsilon^{k-j} \binom{d-j}{d-k} \frac{\kappa_{d-j}}{\kappa_{n-j}} \Phi_j(K, A).$$
Hint: For $B \in \mathcal{B}(\mathbb{R}^d)$ and $\varepsilon > 0$, let
$$U'_\varepsilon(K, B) := \left\{x \in \mathbb{R}^d : 0 < \|x - p_K(x)\| \leq \varepsilon, p_K(x) \in B\right\},$$
where $p_K(x)$ denotes the metric projection of x onto K. First, deduce from the Steiner formula that
$$\lambda_d(U'_\varepsilon(K, B)) = \sum_{j=0}^{d-1} \varepsilon^{d-j} \kappa_{d-j} \Phi_j(K, B).$$
Then show that, for $\varepsilon, \delta > 0$,
$$U'_{\varepsilon+\delta}(K, A) = U'_\varepsilon(K, A) \cup U'_\delta(K + \varepsilon B^d, A + \varepsilon S^{n-1})$$
and observe that this union is disjoint. Use these facts to prove the above formula.

2. Let $k \leq d-1$ and P be a k-dimensional convex polytope in $\mathbb{R}^d$. Then, for any j-face $F \in \mathcal{F}_j(P)$ of P ($j \in \{0, \dots, k\}$), the outer angle $\gamma(F, P)$ of P in F is well defined. The k-dimensional polytope P can also be interpreted as a full dimensional polytope in its (k-dimensional) affine hull $M := \text{aff}P$ and any j-face F of P with respect to $\mathbb{R}^d$ is still a j-face of P with respect to M (where, for $j = k$, P itself is interpreted as the unique k-face of P with respect to M). Let $\gamma'(F, P)$ denote the outer angle of P in F with respect to M, i.e. the corresponding normal cone is now a subset of M.
   (a) Prove that $\gamma(F, P) = \gamma'(F, P)$ for all $F \in \mathcal{F}(P)$, i.e. show that outer angles are independent of the dimension of the ambient space in which P is embedded.
   (b) Use (a) to prove that the curvature measures $\Phi_j(P, \cdot)$ and the intrinsic volumes $V_j(P)$ of a convex polytope P are independent of the dimension of the ambient space.

   Hint: For the proof in (a) you might find the following formula useful: For $n, m \in \mathbb{N}$,
$$\int_{B^m} (1 - \|x\|^2)^{\frac{n}{2}} d\lambda_m(x) = \frac{\kappa_{m+n}}{\kappa_n}.$$

3. Determine the intrinsic volumes $V_j(K)$, $j = 0, \dots, d$, of the convex body K, where
   (a) $K = B^d$ is the d-dimensional unit ball in $\mathbb{R}^d$.
   (b) $K = [0, 1]^d$ is the d-dimensional unit cube and $d = 1, 2, 3$.