Universität Karlsruhe
Institut für Algebra und Geometrie

Prof. Dr. Wolfgang Weil
Dr. Steffen Winter

Stochastic and Integral Geometry I (WS 06/07)

Exercise Sheet 10

Please, hand in your solutions at the end of the **lecture** on **Monday, Jan 29th**!

1. The normal cone $N(P,F)$ of a polytope P at its face F is just a special case of the more general *normal cone* $N(K,x)$ of a convex body $K \subset \mathbb{R}^d$ at a point $x \in K$. It is defined as the set of all outer normals of support hyperplanes of K at x, i.e.
$$N(K,x) := \{v \in \mathbb{R}^d : \langle v, y-x\rangle \leq 0 \text{ for all } y \in K\}.$$
Let K and M be convex bodies in $\mathbb{R}^d$.
   (a) Prove that, if $\text{relint}K \cap \text{relint}M \neq \varnothing$, then, for $x \in K \cap M$, the relation
   $$N(K \cap M, x) = N(K,x) + N(M,x)$$
   holds.
   (b) Show that for $x \in K$ and $y \in M$,
   $$N(K+M, x+y) = N(K,x) \cap N(M,y).$$

2. Prove the following generalization of the Principal kinematic formula for intrinsic volumes: If $K_0, K_1, \dots, K_k$ ($k \in \mathbb{N}$) are convex bodies in $\mathbb{R}^d$ and $j \in \{0,\dots,d\}$, then
$$\int_{G_d} \cdots \int_{G_d} V_j(K_0 \cap g_1K_1 \cap \ldots \cap g_kK_k)\, d\mu(g_1)\ldots d\mu(g_k)$$
$$= \sum_{\substack{m_0,m_1,\ldots,m_k=j \\ m_0+m_1+\ldots+m_k=kd+j}}^{d} c^{m_0,m_1,\ldots,m_k}_{j,\underbrace{d,\ldots,d}_{k-\text{times}}} V_{m_0}(K_0)V_{m_1}(K_1)\ldots V_{m_k}(K_k),$$
where the constants are given (as in the lecture) by
$$c^{m_0,m_1,\ldots,m_k}_{n_0,n_1,\ldots,n_k} = \frac{m_0!\kappa_{m_0}\ldots m_k!\kappa_{m_k}}{n_0!\kappa_{n_0}\ldots n_k!\kappa_{n_k}}.$$

3. Let B be the unit ball in $\mathbb{R}^2$. For a convex body K in $\mathbb{R}^2$, let
$$T(B,K) := \{t \in \mathbb{R}^2 : (K+t) \cap B \neq \varnothing\}$$
be the set of all translations that bring K into a position where it intersects B. A *uniform random translation* on $T(B,K)$ is a random variable $\tilde{t}$ on $T(B,K)$ with distribution
$$\mathbb{P} = \frac{\lambda_2|_{T(B,K)}}{\lambda_2(T(B,K))}.$$
Let $r_1, r_2 > 0$ and $\tilde{t}_1$ and $\tilde{t}_2$ be independent uniform random translations on $T(B, r_1B)$ and $T(B, r_2B)$, respectively. Compute the probability of the event that the two balls $r_1B + \tilde{t}_1$ and $r_2B + \tilde{t}_2$ have non-empty intersection inside B. Determine the explicit value of this probability for the case $r_1 = r_2 = 1$.