Universität Karlsruhe (TH)
Institut für Algebra und Geometrie

Prof. Dr. Wolfgang Weil
Mario Hörig

Stochastic and Integral Geometry II

Exercise Sheet 1

Please hand in your solutions for exercises 3 and 4 at the end of the lecture on **monday, april 30th**. Exercises 1 and 2 will be discussed in the first exercise class on friday, april 27th and thus are not to be solved at home as indicated by * !

Exercise 1*

Let E be a locally compact space with a countable base. Show that the following assertions hold:

a) The topology of E has a countable base $\mathcal{D}$ consisting of open, relatively compact sets such that every open set $G \subset E$ is the union of the sets $D \in \mathcal{D}$ satisfying $\operatorname{cl} D \subset G$.

b) There is a sequence $(G_i)_{i\in\mathbb{N}}$ of open, relatively compact sets in E satisfying $\operatorname{cl} G_i \subset G_{i+1}$ for $i \in \mathbb{N}$ and $\cup_{i\in\mathbb{N}} G_i = E$.

c) For every compact set $C \subset E$ there exists a decreasing sequence $(G_i)_{i\in\mathbb{N}}$ of open, relatively compact neighborhoods of C in E such that to every open set $G \subset E$ with $C \subset G$ there is an $i \in \mathbb{N}$ with $G_i \subset G$. Further, there is a decreasing sequence $(H_i)_{i\in\mathbb{N}}$ of open, relatively compact sets with $\operatorname{cl} H_{i+1} \subset H_i$ and $\cap_{i\in\mathbb{N}} H_i = C$.

d) If $C \subset E$ is compact and $G_1, G_2 \subset E$ are open sets with $C \subset G_1 \cup G_2$ then there are compact sets $C_1 \subset G_1$ and $C_2 \subset G_2$ with $C = C_1 \cup C_2$.

Exercise 2*

The space N of (simple, locally finite) counting measures is supplied with the σ-algebra $\mathcal{N}$ generated by the system

$$\mathcal{F} = \{N_{B,k} :\ k \in \mathbb{N}_0,\ B \text{ Borel subset of } E\}$$

where

$$N_{B,k} = \{\eta \in \mathsf{N} : \eta(\mathsf{B}) = \mathsf{k}\}.$$

Show that already the subfamily

$$\mathcal{F}_0 = \{N_{B,0} :\ B \text{ Borel subset of } E\}.$$

generates $\mathcal{N}$ and, moreover, that $\mathcal{F}_0$ is intersection stable.

Exercise 3

Let X be a Poisson process on E with intensity measure Θ and $A_1, \ldots, A_m$, $m \in \mathbb{N}$ pairwise disjoint Borel subsets of E contained in a Borel set A.
Calculate $\mathbb{P}(X(A_1) = k_1, \ldots, X(A_m) = k_m | \ X(A) = j)$ for j, $k_1, \ldots,\ k_m, \in \mathbb{N}_0$.

Exercise 4

Let X be a point process on E with atom-free intensity measure Θ. Show that X is a Poisson process if

$$\mathbb{E} \prod_{x \in X} f(x) = \exp\big(\int_E (f(x) - 1) \, d\Theta(x) \big)$$

holds, for all measurable functions $f : E \to [0, 1]$.