Universität Karlsruhe (TH)
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Stochastic and Integral Geometry II

Exercise Sheet 5

Please hand in your solutions at the end of the lecture on **Tuesday, May 29th**.

Exercise 1

For a polytope $P \in \mathcal{P}$ let $\mathcal{F}(P)$ be the collection of all faces of P. Show that

$$\begin{aligned} \mathcal{P} &\to \mathsf{N}(\mathcal{K}') \\ P &\mapsto \sum_{F \in \mathcal{F}(P)} \delta_F \end{aligned}$$

is measurable with respect to the σ-algebras $\mathcal{B}(\mathcal{P})$ and $\mathcal{N}(\mathcal{K}')$.

Exercise 2

Let m be a face-to-face mosaic and $\mathcal{F}(m) = \bigcup_{P \in m} \mathcal{F}(P)$.

a) Show that

$$F,\, F' \in \mathcal{F}(m) \Rightarrow F \cap F' = \emptyset \text{ or } F \cap F' \in \mathcal{F}(F) \cap \mathcal{F}(F').$$

b) If m is normal and $0 \leq j \leq k \leq d$, show that every j-face of m belongs to

$$\binom{d-j+1}{k-j}$$

k-faces.

Exercise 3

Let M be a locally compact space with countable base and X a point process on $\mathbb{R}^d \times M$ which is stationary in the sense, that $\mathbb{P}_X$ is invariant under all mappings

$$T_x : (y, m) \mapsto (y + x, m), \quad x \in \mathbb{R}^d.$$

Assume that the intensity measure Θ of X fulfills $\Theta(C \times M) < \infty$ for all $C \in \mathcal{C}$, and $\Theta \not\equiv 0$.

a) Show that there exists a constant $\gamma \in (0, \infty)$ and a probability measure $\mathbb{Q}$ on M such that

$$\Theta = \gamma \lambda_d \otimes \mathbb{Q}.$$

b) Show that for measurable $f : \mathbb{R}^d \times M \to [0, \infty)$

$$\mathbb{E} \sum_{(x,m)\in X} f(x, m) = \gamma \int_M \int_{\mathbb{R}^d} f(x, m) d\lambda(x) d\mathbb{Q}(M).$$

X is called a *marked point process*, γ is its *intensity* and $\mathbb{Q}$ is called the *mark distribution.*