Basic Notions Of Topology

Topological Spaces, Bases , Induced Topologies

Let T be an arbitrary set. A system $\mathcal{O}$ of subsets of T is called a *topology* on T, if the following holds:

a) The union of every class of sets in $\mathcal{O}$ is a set in $\mathcal{O}$, i.e. for an arbitrary index set I we have:

$$O_i \in \mathcal{O} \text{ for all } i \in I \implies \bigcup_{i \in I} O_i \in \mathcal{O}.$$

b) The intersection of every finite class of sets in $\mathcal{O}$ is a set in $\mathcal{O}$, i.e. for all $n \in \mathbb{N}$ we have:

$$O_1, \ldots, O_n \in \mathcal{O} \implies \bigcap_{i=1}^{n} O_i \in \mathcal{O}.$$

c) $T, \emptyset \in \mathcal{O}$.

A *topological space* is a pair $(T, \mathcal{O})$ consisting of a set T and a topology $\mathcal{O}$ on T. The sets in $\mathcal{O}$ are called the *open sets*; the complement of an open subset of T is called *closed*.

A system $\mathcal{B}$ of open subsets of a topological space $(T, \mathcal{O})$ ist called an *open base* of the topology, if every open set $O \in \mathcal{O}$ is the union of sets from $\mathcal{B}$, i.e. for all $O \in \mathcal{O}$ holds:

$$\forall\, x \in \mathcal{O}\, \exists\, B \in \mathcal{B} : B \subset O \text{ and } x \in B.$$

An open base is called *countable*, if it only consists of countably many sets.

A topological space $(T, \mathcal{O})$ is called *second countable* if its topology has a countable base. It follows immediately that second countable spaces are *separable*.

If $U \subset T$ is a subset of a topological space $(T, \mathcal{O})$, then the class $\mathcal{O}_U := \{O \cap U \mid O \in \mathcal{O}\}$ of subsets of U forms a topology on U, the so-called *induced topology*. $(U, \mathcal{O}_U)$ is usually referred to as a *subspace* of T.

If $\mathcal{O}$ and $\mathcal{O}'$ both are topologies on the same set T, we say that $\mathcal{O}$ is *finer* than $\mathcal{O}'$, if $\mathcal{O}' \subset \mathcal{O}$. $\mathcal{O}'$ on the other hand is then said to be *coarser* than $\mathcal{O}$.

Let $A \subset T$ be be a subset of T:

(i) The *interior* of A is the largest open set contained in A and is denoted by $\operatorname{int} A$, i.e.

$$\operatorname{int} A = \bigcup_{B \subset A,\, B \in \mathcal{O}} B.$$

(ii) The *closure* of A is the smallest closed set containing A and is denoted by $\operatorname{cl} A$, i.e.

$$\operatorname{cl} A = \bigcap_{B \supset A,\, B \text{ closed}} B.$$

(iii) The *boundary* of A is the closure of A without the interior of A, i.e.

$$\operatorname{bd} A = \operatorname{cl} A \setminus \operatorname{int} A.$$

Neighbourhoods, Convergence, Haussdorff Spaces

Let $(T, \mathcal{O})$ be a topological space and $x \in T$. A subset $N \subset T$ is called a *neighbourhood of* x, if there exists an open set $O \in \mathcal{O}$ such that $O \subset N$ and $x \in O$.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space $(T, \mathcal{O})$ is said to converge to $x \in T$, if for every neighbourhood U of x there exists a $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$.

Let $(T, \mathcal{O})$ have a countable base. Then the following are equivalent:

a) $A \subset T$ is closed.

b) For every sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \to a$ we have $a \in A$.

A *Hausdorff space* is a topological space in which each pair of distinct points can be separated by disjoint neighbourhoods.

Continuous Mappings

A mapping $f : T \to S$ between two topological spaces $(T, \mathcal{O}_1)$ and $(S, \mathcal{O}_2)$ is called *continuous* if the preimage of every open set in S is an open set in T, i.e.

$$f : T \to S \text{ continuous} \quad :\Longleftrightarrow \quad \forall\, O \in \mathcal{O}_2 : f^{-1}(O) \in \mathcal{O}_1.$$

Let $(T, \mathcal{O})$ have a countable base. Then $f : T \to S$ is continuous if and only if for all $x \in T$ holds:

$$\text{For all } (x_n)_{n\in\mathbb{N}} \text{ with } x_n \to x \text{ we have } f(x_n) \to f(x).$$

Compactness

A topological space $(T, \mathcal{O})$ is called *compact*, if every open cover of T has a finite subcover, i.e. for every index set I we have:

$$\bigcup_{i\in I} O_i = T \text{ and } O_i \in \mathcal{O} \text{ for all } i \in I \quad \Longrightarrow \quad \exists\, I' \subset I : |\, I' \,| < \infty \text{ and } \bigcup_{i\in I'} O_i = T.$$

We call a subset $A \subset T$ of T compact, if the subspace $(A, \mathcal{O}_A)$ is a compact.

$(T, \mathcal{O})$ is said to be *locally compact*, if it is a Hausdorff space and for each element $x \in T$ there is a compact neighbourhood of x.

A subset $A \subset T$ of T is called *relatively compact*, if its closure is compact.

Every compact Hausdorff space $(T, \mathcal{O})$ with a countable base is *sequential compact*, i.e. every sequence $(x_n)_{n\in\mathbb{N}}$ in T has a convergent subsequence.

Metric Spaces and Metrizability

A set T is called a *metric space* if there exists a mapping $d : T \times T \to [0, \infty)$ with the following properties:

a) $d(x, y) = 0 \quad \Longleftrightarrow \quad x = y$,

b) $d(x, y) = d(y, x)$ for all $x, y \in T$,

c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in T$.

We say that a subset $O \subset T$ is open if for every $x \in T$ there is an $r > 0$ such that

$$B(x, r) := \{y \in T \mid d(x, y) < r\} \subset O.$$

The class $\mathcal{O} := \{O \subset T \mid O \text{ open}\}$ of all open subsets of T forms a topology on T, usually called the topology *generated by* d.

A *metrizable* space is a topological space $(T, \mathcal{O})$ with the property that there exists a metric d on the set T whose generated topology is precisely $\mathcal{O}$.

Both compact Hausdorff spaces and locally compact spaces are metrizable if and only if they have a countable base.