HOMEWORK 2

(1) Let $M = \mathbb{R}^n$, let $V = \nabla \operatorname{dist}(0, .)$ denote the gradient of the distance function from the origin, and let $\gamma(t) = t\frac{\partial}{\partial x_n}$.
   (a) Prove that $V = \sum_{i=1}^n \frac{x_i}{r}\frac{\partial}{\partial x_i}$, where $r^2 = \sum_{i=1}^n x_i^2$.
   (b) If $A(t) = \nabla V|_{\gamma(t)}$, prove that with respect to the basis $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\}$, $A(t) = \frac{1}{t}I$.

(2) Let M_κ be a space of constant curvature κ, let $\gamma : \mathbb{R} \to M_\kappa$ be a geodesic, and let $A(t)$ be a solution of the Riccati equation $A' + A^2 + R = 0$.
   (a) Suppose that $A(t) = a(t)I$, that is, the equidistant hypersurfaces defining $A(t)$ are *umbilical* along γ. Prove that:
      (i) The function $a(t)$ solves the equation $a' + a^2 + \kappa = 0$.
      (ii) If $\kappa = 1$ then $a(t) = \cot(t - t_0)$ for some t_0.
      (iii) If $\kappa = 0$ then $a(t) = \frac{1}{t-t_0}$ for some t_0 or $a(t) = 0$.
      (iv) If $\kappa = -1$ then $a(t) = \coth(t - t_0)$ or $a(t) = \tanh(t - t_0)$ for some t_0, or $a(t) = \pm 1$.
   (b) Prove that for a generic $A(t)$, there exists a basis $E_1(t), \ldots E_{n-1}(t)$ of parallel vector fields with respect to which $A(t)$ can be written as

$$A(t) = \operatorname{diag}(a_1(t), \ldots a_{n-1}(t))$$

   where each $a_i(t)$ is a solution of $a_i'(t) + a_i^2 + \kappa = 0$.

(3) Let M^n be a Riemannian manifold, $p \in M$, and let $V = \nabla \operatorname{dist}(p, .)$. Let $\exp_p : B_r(0) \subseteq T_pM \to B_r(p) \subseteq M$ be the exponential map around p. We want to show that $A(t) = \nabla V|_{\gamma(t)}$ satisfies $A \sim \frac{1}{t}I$ around $t = 0$.
   (a) Let $e_1, \ldots e_n \in T_pM$ be an orthonormal basis of T_pM and let $x_1, \ldots x_n$ be the coordinate functions on T_pM defined by

$$x_i(v) = g_p(e_i, v).$$

   Prove that the vector fields $X_1, \ldots X_n$ in $B_r(p)$ defined by $X_i = (\exp_p)_* \frac{\partial}{\partial x_i}$ satisfy $\nabla_{X_i} X_j(p) = 0$.
   (b) Let $\operatorname{dist}(0, .) = \exp^{-1}(\operatorname{dist}(p, .))$. Prove that for any $v \in B_r(0)$, $\operatorname{dist}(0, v) = \|v\|$, and therefore $(\exp_p)_*^{-1}(V) = \nabla \operatorname{dist}(0, .) = \sum_{i=1}^n \frac{x_i}{r}\frac{\partial}{\partial x_i}$.
   (c) Choose $e_1, \ldots, e_n$ of T_pM in such a way that $\gamma'(0) = e_n$. Show that $(\exp_p)^{-1}\gamma(t) = t\frac{\partial}{\partial x_n}$.

(d) By letting g' denote the pullback metric on $B_r(0)$, the exponential map becomes an isometry and we can compute $A(t) = \nabla V|_{\gamma(t)}$ directly on T_pM. Show that $A(t) = \frac{1}{t}I + B$, where B depends linearly on the Christoffel symbols $\Gamma_{ij}^k = g'\left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right)$ and in particular $B(t) \sim 0$ around $t = 0$.