HOMEWORK 3

(1) We aim to prove Bonnet–Myers theorem as an application of Rauch I.

Theorem (Bonnet–Myers theorem)**.** *Let M be a complete connected Riemannian manifold with sectional curvature bounded below by K. Then M is compact and $\pi_1(M)$ is finite.*

We prove the Theorem in steps:

(a) Use Rauch I to prove that M is compact and has diameter $\leq \pi/\sqrt{K}$,

(b) Repeat the first point for the universal cover $\tilde{M}$ of M to argue that the fundamental group $\pi_1(M)$ is finite.

(2) We aim to prove Klingenberg's Long Homotopy Lemma:

Theorem (Klingenberg's Lemma)**.** *Let M be a complete Riemannian manifold with sectional curvature $K \leq K_0$, where K_0 is a positive constant. Let $p, q \in M$ and let γ_0 and γ_1 be two distinct geodesics joining p to q with $L(\gamma_0) \leq L(\gamma_1)$. Assume that γ_0 is homotopic to γ_1, that is, there exists a continuous family of curves α_t, $t \in [0,1]$ such that $\alpha_0 = \gamma_0$ and $\alpha_1 = \gamma_1$. Then there exists $t_0 \in [0,1]$ such that*

$$L(\gamma_0) + L(\alpha_{t_0}) \geq \frac{2\pi}{\sqrt{K_0}}.$$

We prove the theorem in steps:

(a) By using Rauch's Theorem prove that $\exp_p : T_pM \to M$ has no critical points in the open ball B of radius $\frac{\pi}{\sqrt{K_0}}$.

(b) Show that for small t, there exists a curve $\tilde{\alpha}_t$ in T_pM, joining $\exp_p^{-1}(p) = 0$ to $\exp_p^{-1}(q) = \tilde{q}$, such that $\exp_p \tilde{\alpha}_t = \alpha_t$.

(c) Prove that α_1 can not be lifted keeping the endpoints fixed.

(d) Using contradiction method show that for all $\epsilon > 0$, there exists a $t(\epsilon)$ such that $\alpha_{t(\epsilon)}$ can be lifted to $\tilde{\alpha}_{t(\epsilon)}$ and $\tilde{\alpha}_{t(\epsilon)}$ contains points with distance $< \epsilon$ from the boundary of ∂B of B.

(e) For all $\epsilon > 0$, show that

$$L(\gamma_0) + L(\alpha_{t(\epsilon)}) \geq \frac{2\pi}{\sqrt{K_0}} - 2\epsilon,$$

and then get the inequality in the Lemma.

(3) We prove the following corollary of the Algebraic Comparison Theorem

Theorem (Weak Algebraic Comparison)**.** *Let* $R_1, R_2 : \mathbb{R} \to \mathrm{Sym}^2(\mathbb{R}^{n-1})$ *be such that* $\lambda_+(R_2) \leq \lambda_-(R_1)$. *For* $i = 1, 2$, *let* $A_i : [t_0; t_i) \to \mathrm{Sym}^2(\mathbb{R}^{n-1})$ *be a solution of*

$$A_i' + A_i^2 + R_i = 0$$

with maximal $t_i \in (0, \infty]$. *Assume that* $\lambda_+(A_1(t_0)) \leq \lambda_-(A_2(t_0))$. *Then* $t_1 \leq t_2$ *and* $\lambda_+(A_1) \leq \lambda_-(A_2)$ *on* (t_0, t_1).

We prove this in steps:

(a) Let $A, B \in \mathrm{Sym}^2(\mathbb{R}^{n-1})$. Show that $\lambda_+(A) \leq \lambda_-(B)$ iff for every $X \in SO(n-1)$, $XAX^T \leq B$.

(b) Let $R_i : \mathbb{R} \to \mathrm{Sym}^2(\mathbb{R}^{n-1})$ and A_i be a solution of

$$A_i' + A_i^2 + R_i = 0,$$

and let $\bar{R}_i = X^T R_i X$, $\bar{A}_i = X^T A_i X$. Show that $\bar{A}_i$ solves the equation

$$\bar{A}_i' + \bar{A}_i^2 + \bar{R}_i = 0.$$

(c) Use the Algebraic Comparison Theorem, together with the two points above, to prove the Weak Algebraic Comparison.