(1) We aim to prove Bonnet–Myers theorem as an application of Rauch I.

**Theorem** (Bonnet–Myers theorem). Let $M$ be a complete connected Riemannian manifold with sectional curvature bounded below by $K$. Then $M$ is compact and $\pi_1(M)$ is finite.

We prove the Theorem in steps:
(a) Use Rauch I to prove that $M$ is compact and has diameter $\leq \pi/\sqrt{K}$,
(b) Repeat the first point for the universal cover $\tilde{M}$ of $M$ to argue that the fundamental group $\pi_1(M)$ is finite.

(2) We aim to prove Klingenberg’s Long Homotopy Lemma:

**Theorem** (Klingenberg’s Lemma). Let $M$ be a complete Riemannian manifold with sectional curvature $K \leq K_0$, where $K_0$ is a positive constant. Let $p,q \in M$ and let $\gamma_0$ and $\gamma_1$ be two distinct geodesics joining $p$ to $q$ with $L(\gamma_0) \leq L(\gamma_1)$. Assume that $\gamma_0$ is homotopic to $\gamma_1$, that is, there exists a continuous family of curves $\alpha_t$, $t \in [0,1]$ such that $\alpha_0 = \gamma_0$ and $\alpha_1 = \gamma_1$. Then there exists $t_0 \in [0,1]$ such that

$$L(\gamma_0) + L(\alpha_t) \geq \frac{2\pi}{\sqrt{K_0}}.$$

We prove the theorem in steps:
(a) By using Rauch’s Theorem prove that $\exp_p : T_p M \to M$ has no critical points in the open ball $B_\pi(\frac{\pi}{\sqrt{K_0}})$.
(b) Show that for small $t$, there exists a curve $\tilde{\alpha}_t$ in $T_p M$, joining $\exp_p^{-1}(p) = 0$ to $\exp_p^{-1}(q) = \tilde{q}$, such that $\exp_p \tilde{\alpha}_t = \alpha_t$.
(c) Prove that $\alpha_1$ can not be lifted keeping the endpoints fixed.
(d) Using contradiction method show that for all $\epsilon > 0$, there exists a $t(\epsilon)$ such that $\alpha_{t(\epsilon)}$ can be lifted to $\tilde{\alpha}_{t(\epsilon)}$ and $\tilde{\alpha}_{t(\epsilon)}$ contains points with distance $< \epsilon$ from the boundary of $\partial B$ of $B$.
(e) For all $\epsilon > 0$, show that

$$L(\gamma_0) + L(\alpha_{t(\epsilon)}) \geq \frac{2\pi}{\sqrt{K_0}} - 2\epsilon,$$

and then get the inequality in the Lemma.

(3) We prove the following corollary of the Algebraic Comparison Theorem
Theorem (Weak Algebraic Comparison). Let $R_1, R_2 : \mathbb{R} \to \text{Sym}^2(\mathbb{R}^{n-1})$ be such that $\lambda_+(R_2) \leq \lambda_-(R_1)$. For $i = 1, 2$, let $A_i : [t_0; t_i) \to \text{Sym}^2(\mathbb{R}^{n-1})$ be a solution of
\[ A'_i + A^2_i + R_i = 0 \]
with maximal $t_i \in (0, \infty]$. Assume that $\lambda_+(A_1(t_0)) \leq \lambda_-(A_2(t_0))$. Then $t_1 \leq t_2$ and $\lambda_+(A_1) \leq \lambda_-(A_2)$ on $(t_0, t_1)$.

We prove this in steps:
(a) Let $A, B \in \text{Sym}^2(\mathbb{R}^{n-1})$. Show that $\lambda_+(A) \leq \lambda_-(B)$ iff for every $X \in \text{SO}(n-1)$, $XAX^T \leq B$.
(b) Let $R_i : \mathbb{R} \to \text{Sym}^2(\mathbb{R}^{n-1})$ and $A_i$ be a solution of
\[ A'_i + A^2_i + R_i = 0, \]
and let $\bar{R}_i = X^T R_i X$, $\bar{A}_i = X^T A_i X$. Show that $\bar{A}_i$ solves the equation
\[ \bar{A}'_i + \bar{A}^2_i + \bar{R}_i = 0. \]
(c) Use the Algebraic Comparison Theorem, together with the two points above, to prove the Weak Algebraic Comparison.