HOMEWORK 5'

(1) Prove the following version of Toponogov Comparison Theorem.

Definition. A hinge in a complete Riemannian manifold consists of two geodesic segments γ_1, γ_2 such that $\gamma_1(0) = \gamma_2(0)$. We denote it by $(\gamma_1, \gamma_2, \alpha)$, where $\alpha = \angle(\gamma_1'(0), \gamma_2'(0))$.

Theorem. *(Hinge Version of Toponogov Theorem) Let M be a complete Riemannian manifold with $sec \geq k$, and let $\triangle xyz$ be a triangle in M with sides $\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$, and the hinge $(\gamma_{xy}, \gamma_{yz}, \alpha)$. Let $\triangle \tilde{x}\tilde{y}\tilde{z}$ be a triangle in $\tilde{M}_k$ with sides $\gamma_{\tilde{x}\tilde{y}}, \gamma_{\tilde{y}\tilde{z}}, \gamma_{\tilde{z}\tilde{x}}$, and the hinge $(\gamma_{\tilde{x}\tilde{y}}, \gamma_{\tilde{y}\tilde{z}}, \tilde{\alpha})$ such that $\ell(\gamma_{xy}) = \ell(\gamma_{\tilde{x}\tilde{y}})$, $\ell(\gamma_{yz} = \gamma_{\tilde{y}\tilde{z}})$, and $\alpha = \tilde{\alpha}$, then $\ell(\gamma_{zx}) \leq \ell(\gamma_{\tilde{z}\tilde{x}})$.*

[Hint: Assume by contradiction that $\ell(\gamma_{zx}) \leq \ell(\gamma_{\tilde{z}\tilde{x}})$, and use the Corollary of Toponogov Theorem to obtain a contradiction].

(2) Prove Rauch Comparison Theorem I by using the hinge version of Toponogov, and contradiction method.

(3) In the following we aim to prove an upper bound on the perimeter of triangles in complete Riemannian manifolds with curvature bounded below by a positive number.
   (a) Show that every triangle on the 2-sphere, with perimeter 2π and sides with length $< \pi$ must lie on a great circle. Conclude that its angles are π.
   (b) Show that for any two shortest geodesics γ_{xz}, γ_{yw}, where y is an inner point of γ_{xz}, one has $\angle xyw + \angle wyz = \pi$.
   (c) Let M be a complete Riemannian manifold with $sec \geq 1$, $k > 0$, and let $\operatorname{diam} M < \pi$. Prove that there exists no triangle in M with perimeter greater than 2π.
   (d) Let M be a complete Riemannian manifold with $sec \geq 1$, then prove that there exists no triangle in M with perimeter greater than 2π.