HOMEWORK 6

(1) (a) Let M be a Riemannian manifold , and $p \in \widetilde{M}$ be a point in the universal cover of M. Then prove that the function $\pi_1(M) \to \mathbb{R}$ given by $g \mapsto d(p, gp)$ has a discrete image, i.e., the image does not have accumulation points.

(b) In the proof of Gromov's Theorem on the generators of $\pi_1(M)$ (Theorem 7.3 of Eschenburg's Note), show that the set $\Gamma \setminus \Gamma_k$ does indeed have an element of minimal norm.

(2) Let Γ be a finitely generated discrete group, and let $N(k)$ denote the growth function of Γ with respect to some set of generators. Recall that Γ has *polynomial growth* of order $\geq p$, resp. $\leq p$, if

$$\liminf_{k\to\infty} \frac{N(k)}{k^p} > 0, \qquad resp. \quad \limsup_{k\to\infty} \frac{N(k)}{k^p} < \infty.$$

Let M be a complete Riemannian manifold. In analogy to the above, M has *polynomial volume growth* of order $\geq p$, resp. $\leq p$, if the above equations hold with

$$\liminf_{r\to\infty} \frac{VolB_r(p)}{r^p} > 0, \qquad resp. \quad \limsup_{r\to\infty} \frac{VolB_r(p)}{r^p} < \infty.$$

We aim to prove the following.

Theorem. *Let M be a complete Riemannian manifold of polynomial volume growth of order $\geq k$. Let $\pi : \widetilde{M} \to M$ be a normal covering, with covering group Γ a finitely generated discrete group. If the volume growth of $\widetilde{M}$ is polynomial of order $\leq k + l$, then Γ has polynomial growth of order $\leq k$.*

(a) Show that π maps $B_r^{\widetilde{M}}(\tilde{o}) \cap F$ onto $B_r^M(o)$, where F is a fundamental domain (cf. Eschenburg's notes), and $\pi(\tilde{o}) = o$.

(b) Using the fact that $F \setminus Int(F)$ has Lebesque measure zero, show that

$$Vol(B_r^M(o)) = Vol(B_r^{\widetilde{M}}(\tilde{o}) \cap IntF)$$

(c) Define $\Omega_g = B_r^{\widetilde{M}}(g\tilde{o}) \cap g\, Int(F)$. Show that Ω_g's are disjoint, and for a constant c,

$$\bigcup_{g\in U(r)} \Omega_g \subseteq B_cr^{\widetilde{M}}(\tilde{o}).$$

(d) Prove the theorem.

(3) (a) Let M be a complete Riemannian manifold with $Ric \geq 0$, and G be a finitely generated subgroup of $\pi_1(M)$. Show that $M' = \widetilde{M}/G$ and $N' = \widetilde{M'}$ satisfy all conditions of the theorem of Problem 2 with $k = 0$, $l = n$.
(b) Prove Milnor theorem using part (a)

(4) (a) Show that the group $\mathbb{Z} * \mathbb{Z}$ has exponential growth.
(b) Show that $M = (S^n \times S^1)\#(S^n \times S^1)$ does not admit a metric with nonnegative Ricci curvature.

Hint. *Show that the fundamental group of M is $\mathbb{Z} * \mathbb{Z}$.*

(c) Prove that any surface of genus $g \geq 2$ does not admit a metric with nonnegative Ricci curvature.

Hint. *It is enough to prove the problem for a surface of genus 2.*