HOMEWORK 7

(1) Let M be a Riemannian manifold.

(a) Take $p \in M$, and suppose R satisfies

$$R \leq \frac{1}{2}\frac{\pi}{\sqrt{\kappa}}, \quad \kappa = \sup\{\sec(\Pi) \mid \Pi \subseteq T_x M \, x \in B_R(p)\}.$$

Prove that $B_R(p)$ is convex (Hint: distance balls are sub level sets of distance functions. Show that such distance functions are, locally, convex).

(b) Using the point above, prove that for any $p \in M$, $B_r(p)$ is convex if $r < \frac{1}{2}\frac{\pi}{\sqrt{\kappa}}$, with $\kappa = \sup\{\sec(\Pi) \mid \Pi \subset TM\}$ the supremum of all sectional curvatures of M.

(c) Given a family $\{f_\alpha\}_\alpha$ of convex functions, prove that $f = \sup f_\alpha$ is convex as well. Similarly, if $\{g_\alpha\}_\alpha$ is a family of concave functions, prove that $g = \inf g_\alpha$ is concave as well.

(2) Let C be a convex subset of a Riemannian manifold and $p \in C$. Define the *tangent cone* of C at p as

$$T_pC = \{v \in T_pM \mid \exp_p(tv) \in C, \text{ for all } t \in (0, t_0)\}.$$

We aim to show that T_pC is convex in T_pM.

(a) Let $v_1, v_2 \in T_pC$, and $v = (1-s)v_1 + sv_2$. Show that v is uniquely characterized by the equations

$$\|v_1 v\| \leq s\|v_1 v\|$$
$$\|v_2 v\| \leq (1-s)\|v_1 v\|.$$

(b) Let $p_n = \exp_p(\frac{1}{n}v_1)$, $q_n = \exp_p(\frac{1}{n}v_2)$, and σ_n be the minimizing geodesic from p_n to q_n. Also let $w_n \in T_pC$ be such that $\exp_p(\frac{1}{n}w_n) = \sigma_n(s)$. Show that w_n (sub)-converges to $v = (1-s)v_1 + sv_2$ and thus $v \in T_pC$.

Hint. *Use the fact that* $\exp_p|_{B_{\frac{1}{n}}(p)}$ *is* C_n*-bilipschitz, where* $C_n \to 1$ *as* $n \to \infty$*, to prove that the limit of* w_n *satisfies the conditions of point (a).*

(3) Let f be a smooth convex function on a complete Riemannian manifold M. Then

(a) Prove that the critical points of f are its absolute minimum points.

(b) Prove that f is constant on each closed geodesic in M.

Hint. *Use the fact that sublevel sets of f are totally convex.*

(c) For any value c of f, let $M^c = \{p \in M : f(p) \leq c\}$. Prove that the inclusion $M^c \subseteq M$ induces a surjection $\pi_1(M^c) \to \pi_1(M)$, where.

Hint. *Show that each element of $\pi_1(M,p)$, $p \in M^c$, can be represented by a geodesic loop.*

(4) Here we aim to prove the following result:

Theorem (Bishop-O'Neill, Yau)**.** *There is no non-trivial continuous convex function on a complete manifold with finite volume.*

(a) Let f be a continuous convex function on M, and γ be a geodesic in M such that $\lim_{t_i\to\infty}\gamma(t_i) = \gamma(0)$, for some sequence $t_i \to \infty$. Show that f is constant on $\{\gamma(t), t > 0\}$.

(b) Let $S(M)$ be the unit sphere bundle of M, and let F_t be the one-parameter group generated by the geodesic flow. Show that for any open set U in $S(M)$ there is an infinite sequence $t_i \to \infty$ such that $F_{t_i}(U) \cap U \neq \emptyset$, for all t_i.

Hint. *Prove it by contradiction. Use the fact $S(M)$ has finite measure.*

(c) Let $L = \{x \in S(M) : F_{t_i}(x) \to x$, for some sequence $t_i \to \infty\}$. Show that L is a dense subset of $S(M)$.

Hint. *For $U \subseteq S(M)$, let V_1 be an open subset of U with bounded diameter. then $F_{t_1}(V_1) \cap V_1 \neq \emptyset$, for some $t_1 > 1$. Let V_2 be open subset of V_1 such that $V_2' = V_2 \cap F_{t_1}^{-1}(V_1) \neq \emptyset$ and $\operatorname{diam} V_2 \leq \frac{1}{2}\operatorname{diam} V_1$. Then $V_2' \cap F_{t_2}^{-1}(V_2') \neq \emptyset$ for some $t_2 > 2$. Continuing in this way we construct a decreasing sequence of open sets $\{V_i\}$ with $\operatorname{diam}(V_i) \to 0$ which gives us a point x such that $F_{t_i}(x) \to x$.*

(d) Let (y, v) be a point in $S(M)$, and γ_y be a geodesic defined by (y, v). Show that f is constant on γ_y and get the result.

Hint. *Show that this is the case if $(y, v) \in L$, and use the density of L.*

(5) Use the previous exercise to prove that an open Riemannian manifold with nonnegative sectional curvature must have infinite volume.