HOMEWORK 8

(1) Given a geodesic $\gamma : [0, \infty) \to M$ which is not a ray, is the Busemann function b_γ always well defined?

Hint. *Take a geodesic on cylinder which is not a ray.*

(2) Find all the possible souls of $S^1 \times \mathbb{R}^2$ endowed with the product metric, using the algorithm provided. Do the same thing, by replacing the flat metric on $\mathbb{R}^2$ with the paraboloid metric (i.e. the metric on $\mathbb{R}^2$ induced by the immersion into $\mathbb{R}^3$ as the graph of $z = x^2 + y^2$).

(3) Recall that, in the construction of the soul, we produced a filtration $M \supset C_0 \supset C_1 \supset \ldots \supset C_k$ by convex sets.
  (a) Let k be the last term of the sequence given in constructing soul. Show that $k \leq n$.
  (b) Find the diffeomorphism type of M, when $k = n$.

(4) Show that the open manifold $\mathbb{T}^2 \setminus \{p\}$ does not admit a **complete** metric with non-negative sectional curvature. In general: for which orientable surface M_g one can find a complete metric of nonegative sectional curvature on $M_g \setminus \{p\}$?

(5) Here we aim to find the souls of the tangent bundle of a homogeneous space G/H, where G is a compact Lie group and H is a closed subgroup of G. Let $\mathfrak{g}, \mathfrak{h}$ be the corresponding Lie algebras.
  (a) Show that $T(G/H) = (G \times \mathfrak{p})/H$, where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ (with respect to a bi-invariant metric) and $h \cdot (g, v) = (hg, \mathrm{Ad}(h)v)$.
  (b) Prove that the H action is free and isometric, and therefore the projection $G \times_H \mathfrak{p} \to T(G/H)$ induces a metric of non-negative sectional curvature on $M = T(G/H)$.
  (c) Prove that a soul of M is the zero section.

  Hint. *Start the construction of the soul with a point on the zero section.*