HOMEWORK 4

(1) Let $S = (-1, 1)^{n-1}$ with coordinates $s_1, \ldots, s_{n-1}$. Let 

$$
\gamma : S \times \mathbb{R} \to M
$$

be an $(n - 1)$-parameter variation of geodesics with $S_t = \gamma(S \times \{t\})$ as equidistant hypersurfaces. For all $s \in S$, choose a parallel orthonormal frame $e_1(t), \ldots, e_{n-1}(t)$ along $\gamma_s(t)$, and define $X_i(s, t) = d(s, t)\gamma_s(\partial_{e_i}(s, t))$.

(a) Prove that $J_1(t) = X_1|_{\gamma_s(t)}$, $\ldots$, $J_{n-1}(t) = X_{n-1}|_{\gamma_s(t)}$ are Jacobi fields satisfying $J'_i(t) = A_i J_i(t)$.

(b) For all $s \in S$, let $j_s(t) = \det(\langle J_i(t), e_j(t) \rangle)$. Prove that 

$$
Vol(S_t) = \int_S j_s(t) \, ds_1 \ldots ds_{n-1}.
$$

(2) In the following two problems we will see some applications of the Bishop-Gromov volume comparison theorem. Here, we aim to find a lower bound of the volume growth.

**Theorem (Calabi–Yau).** Let $(M^n, g)$ be a complete non-compact Riemannian manifold with $\text{Ric} \geq 0$. Then there exists a positive constant $c$ depending only on $p$ and $n$ so that 

$$
Vol(B_r(p)) \geq cr
$$

for any $r \geq 2$.

We prove the Theorem in steps:

(a) Show that for any $p \in M$ there exists a geodesic $\gamma : [0, \infty) \to M$ with $\gamma(0) = p$ such that $\text{dist}(p, \gamma(t)) = t$ for $t > 0$.

(b) For any $t > \frac{3}{2}$, use the Bishop-Gromov volume comparison theorem to show that 

$$
\frac{Vol(B_{t+1}(\gamma(t)))}{Vol(B_{t-1}(\gamma(t)))} \leq \frac{(t + 1)^n}{(t - 1)^n}.
$$

(c) Show that $B_1(p) \subset B_{t+1}(\gamma(t)) \setminus B_{t-1}(\gamma(t))$ and it follows 

$$
Vol(B_{t-1}(\gamma(t))) \leq C(n)Vol(B_1(p))t,
$$

where $C(n)$ is the infimum of the function $\frac{1}{t} \frac{(t+1)^n}{(t-1)^n}$ on $[\frac{3}{2}, \infty)$.

(d) Use the fact that $B_r(p) \supset B_{\frac{r+1}{2} - 1}(\gamma(\frac{r+1}{2}))$ to get the theorem.

(3) The rigid version of Bishop-Gromov Theorem reads as follows:
Theorem. Let $M$ be an $n$-dimensional manifold with $\text{Ric} \geq (n-1)\kappa$ and let $\overline{M}_\kappa$ be the simply connected, $n$-dimensional manifold with $\text{sec} \equiv \kappa$. If for some point $p \in M$, $\overline{p} \in \overline{M}_\kappa$ and $r > 0$ we have $\text{Vol}(B_r(p)) = \text{Vol}(B_r(\overline{p}))$ then $B_r(p)$ and $B_r(\overline{p})$ are isometric.

Using Bishop-Gromov Theorem and its rigid version above, we now prove the following theorem, which describes what happens when the result in Bonnet-Myers Theorem is sharp.

Theorem. Let $(M, g)$ be a complete Riemannian manifold with $\text{Ric} \geq (n-1)k$ for some $k > 0$, and $\text{diam}(M, g) = \frac{\pi}{\sqrt{k}}$, then $M$ is isometric to the standard sphere of radius $\frac{1}{\sqrt{k}}$.

(a) For simplicity assume $k = 1$ and use Bishop-Gromov volume comparison theorem to show that for any point $p \in M$,

$$\frac{\text{Vol}(B_{\pi/2}(p))}{\text{Vol}(M)} \leq \frac{1}{2}.$$ 

(b) For $p, q \in M$ such that $d(p, q) = \pi$, show that $B_{\pi/2}(p)$ and $B_{\pi/2}(q)$ are both isometric to half sphere and then get the result.