Institut für Algebra und Geometrie
Dr. F. Galaz-García

Karlsruher Institut für Technologie

Comparison Geometry in Summer 2017
Exercise sheet 6

Exercise 1.
Let M_κ be a Riemannian manifold of constant curvature κ. Let $\gamma\colon \mathbb{R} \to M_\kappa$ be a geodesic and let $A(t) = a(t)I$ be a solution of the Riccati equation $A' + A^2 + R = 0$. Prove the following assertions:

(i) The function $a(t)$ is a solution of the equation $a' + a^2 + \kappa = 0$.

(ii) If $\kappa = 1$, then $a(t) = \cot(t - t_0)$ for some t_0.

(iii) If $\kappa = 0$, then $a(t) = \frac{1}{t-t_0}$ for some t_0 or $a(t) = 0$.

(iv) If $\kappa = -1$, then $a(t) = \coth(t - t_0)$ or $a(t) = \tanh(t - t_0)$ for some t_0, or $a(t) = \pm 1$.

Exercise 2.
Let M be a complete manifold with non-negative sectional curvature. Further, let $p_0, p_1 \in M$ be two points joined by a minimising geodesic $\gamma\colon [0,1] \to M$ and let X be a parallel vector field along γ such that $X \perp \gamma'$.

(i) Show that, for $p_s(t) = \exp tX(s)$ with $s \in [0,1]$,

$$d(p_0(t), p_1(t)) \leq d(p_0, p_1).$$

(ii) Show that equality holds for some $t_0 > 0$ if and only if $p_0, p_1, p_0(t_0), p_1(t_0)$ bound a totally geodesic rectangle.

Due: Wednesday June 14, 2017, before the exercise class.