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Comparison Geometry in Summer 2017
Exercise sheet 7

Exercise 1.
Let (M^{2n}, g) be an orientable and positively curved Riemannian manifold. Let γ be a closed geodesic in M, i.e. $\gamma\colon S^1 \to M$ is an immersion that is a geodesic at all of its points. Show that γ is homotopic to some closed curve c in M with $L(c) < L(\gamma)$.

Exercise 2.
Let M^n be a complete, connected Riemannian manifold of dimension n with positive sectional curvature everywhere. Recall that a submanifold N of M is called *totally geodesic* if the geodesics with respect to the metric in N are geodesics in M. Show that any two compact, totally geodesic submanifolds A, B of M with $\dim A + \dim B \geq n$ must intersect.

Exercise 3.
Suppose that the sectional curvature sec of a Riemannian manifold M satisfies the inequality

$$0 < L \leq K \leq H,$$

where H and L are constants. Let γ be a geodesic in M. Then the distance d along γ between two consecutive conjugate points of γ satisfies

$$\frac{\pi}{\sqrt{H}} \leq d \leq \frac{\pi}{\sqrt{L}}.$$

Exercise 4.
A smooth manifold M is said to have *almost non-positive curvature* if, for every $\varepsilon > 0$, there exists a complete Riemannian metric g_ϵ on M such that $\sec_{g_\varepsilon} \leq K$ and $K \operatorname{diam}(M, g_\varepsilon)^2 \leq \varepsilon$.

Show that S^2 does not have almost non-positive curvature.
Hint: Use the Gauß-Bonnet Theorem.

Due: Wednesday June 28, 2017, before the exercise class.