

Comparison Geometry in Summer 2017 Exercise sheet 10

Exercise 1.

Prove the following theorem.

Theorem (Gromov, 1978). Let (M^n, g) be a complete connected Riemannian manifold with $\text{sec}_g \geq 0$. Then $\pi := \pi_1(M, p_0)$ contains a generating set of $N \leq C(n)$ elements, where $C(n)$ is a number only depending on the dimension n of M .

You may want to proceed as follows.

Define $|l| := d(p, l \cdot p)$ for all $l \in \pi$ and pick l_1 with $|l_1|$ minimal. Inductively define

$$\Gamma_k := \langle l_1, \dots, l_k \rangle \subset \pi$$

with $l_{k+1} \in \pi \setminus \Gamma_k$ with $|l_{k+1}|$ minimal. Now it suffices to show that $\Gamma_k = \pi$ for some $k \leq C(n) := 2\sqrt{5}^n$.

- (i) Consider the triangle $p, p_i := g_i \cdot p, p_j := g_j \cdot p$ for $j > i$ and use Toponogov's theorem to conclude that $\alpha_{ij} \geq 60^\circ$ for all $i \neq j$, where α_{ij} is the angle between v_i and v_j such that $\gamma_{v_i}: p \rightsquigarrow p_i$ and $\gamma_{v_j}: p \rightsquigarrow p_j$ are shortest geodesics joining, respectively, p to p_i and p to p_j .
- (ii) Notice that for any two such vectors v_i, v_j , the balls with radius $\frac{1}{2}$ centered at v_i and v_j are disjoint and their inner half balls are contained in $B_{\frac{\sqrt{5}}{2}}(0)$. Deduce, by comparing the volume of $B_{\frac{\sqrt{5}}{2}}(0)$ with the volume of $B_{\frac{1}{2}}$, that there can be at most $2\sqrt{5}^n$ vectors such that $\alpha_{ij} \geq 60^\circ$.