Karlsruher Institut für Technologie

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Comparison Geometry in Summer 2015

Exercise sheet 2

Exercise 1.
Prove or disprove: Any complete Riemannian metric on $S^1 \times S^2$ must have positive sectional curvature somewhere.

Exercise 2 (Nonpositive curvature I)**.**
Let (M, g) be a simply-connected complete Riemannian manifold with nonpositive sectional curvature. Show that every isometry $\Phi\colon M \to M$ of finite order in $\mathrm{Isom}(M)$, i.e. there exists $n \in \mathbb{N}$ such that $\Phi^n = \mathrm{id}_M$ has a fixed point.

Exercise 3 (Nonpositive curvature II)**.**
Let (M, g) be a complete Riemannian manifold with nonpositive sectional curvature. Show that $\pi_1(M)$ is torsion free, i.e. that there are no finite order elements contained in the fundamental group.

Exercise* 4 (Properties of Jacobi fields)**.**
Let γ be a geodesic on a Riemannian manifold (M, g) and let J be a Jacobi field along γ. Prove the following three assertions:

(a) $\partial_t \langle J(t), \dot\gamma(t)\rangle$ is constant.

(b) $J(0) \perp \dot\gamma(0)$ and $\dot J(0) \perp \dot\gamma(0)$ together imply $J(t) \perp \dot\gamma(t)$ for all t.

(c) $\mathcal{J}_\gamma := \{J \text{ is a Jacobi field along } \gamma\} \cong \mathrm{span}\{\dot\gamma, t\dot\gamma\} \oplus \{J \in \mathcal{J}_\gamma \mid J(t) \perp \dot\gamma(t)\ \forall t\}$.

Exercise 5 (Conjugate points)**.**
Let γ_v be a geodesic emanating from p with velocity $v \in \mathrm{T}_p M$, i.e. $\gamma_v(0) = p$ and $\dot\gamma(0) = v$. Show that v is a critical point of $\exp_p$ if an only if there exists a nontrivial Jacobi field J along γ_v such that $J(0) = J(|v|) = 0$.

Due: Wednesday April 29, 2015, before the exercise class.