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Comparison Geometry in Summer 2015

Exercise sheet 3

Exercise* 1.
Let $N_1, N_2 \subset M$ be closed, disjoint submanifolds of a compact Riemannian manifold (M, g). Show that there is a minimising geodesic $\gamma\colon [0,1] \to M$ between N_1 and N_2 which is perpendicular to both submanifolds, i.e. $\dot\gamma(0) \perp \mathrm{T}_{\gamma(0)} N_1$ and $\dot\gamma(1) \perp \mathrm{T}_{\gamma(1)} N_2$.

Exercise 2.
Let $f\colon (-\varepsilon, \varepsilon) \times [0, a] \to M$ be a variation of a piecewise smooth curve $c\colon [0,a] \to M$ in a Riemannian manifold M. Prove that on any rectangle $(-\varepsilon, \varepsilon) \times [t_i, t_{i+1}]$, where f is smooth, one has

$$\frac{\mathrm{D}}{\mathrm{d}\, s}\frac{\partial f}{\partial t} = \frac{\mathrm{D}}{\mathrm{d}\, t}\frac{\mathrm{d}\, f}{\mathrm{d}\, s}.$$

Exercise 3.
Let M be a Riemannian manifold and let

$$\Omega_{p,q} := \{c\colon [0,1] \to M \mid c \text{ is piecewise smooth and } c(0) = p, c(1) = q\}.$$

Show that a constant speed curve $c \in \Omega_{p,q}$ minimises the arc length functional $L\colon \Omega_{p,q} \to [0, \infty)$ if and only if it minimises the energy functional $E\colon \Omega_{p,q} \to [0, \infty)$.

Exercise 4.
Let (M, g) be a complete simply-connected nonpositively curved Riemannian manifold and let $\lambda\colon \mathbb{R} \to M$ be a geodesic parametrised by arc length. For a point $p \in M \setminus \mathrm{Im}(\lambda)$ define $\mathrm{d}(s) = \mathrm{d}_g(p, \lambda(s))$.

(a) Consider a family of geodesics $\gamma_s\colon [0, \mathrm{d}(s)] \to M$ from p to $\lambda(s)$ and show that

$$\frac{1}{2}E'(s) = \left\langle \lambda'(s), \gamma_s'(\mathrm{d}(s)) \right\rangle.$$

(b) Conclude that s_0 is a critical point of d if and only if $\left\langle \lambda'(s_0), \gamma_{s_0}'(d(s_0)) \right\rangle = 0$.

Exercise 5.
Let M be a complete Riemannian manifold. Let p, q be points in M and let $\gamma\colon [0, a] \to M$ be a minimising geodesic joining p to q. Show that, for all piecewise smooth curves $c\colon [0, a] \to M$ joining p to q,

$$E(\gamma) \leq E(c),$$

with equality holding if and only if c is a minimising geodesic.

Due: Wednesday May 6, 2015, before the exercise class.