Exercise* 1.
Let $N_1, N_2 \subset M$ be closed, disjoint submanifolds of a compact Riemannian manifold $(M, g)$. Show that there is a minimising geodesic $\gamma: [0, 1] \rightarrow M$ between $N_1$ and $N_2$ which is perpendicular to both submanifolds, i.e. $\dot{\gamma}(0) \perp T_{\gamma(0)} N_1$ and $\dot{\gamma}(1) \perp T_{\gamma(1)} N_2$.

Exercise 2.
Let $f: (-\varepsilon, \varepsilon) \times [0, a] \rightarrow M$ be a variation of a piecewise smooth curve $c: [0, a] \rightarrow M$ in a Riemannian manifold $M$. Prove that on any rectangle $(-\varepsilon, \varepsilon) \times [t_i, t_{i+1}]$, where $f$ is smooth, one has
\[
\frac{D}{ds} \frac{\partial f}{\partial t} = \frac{D}{dt} \frac{df}{ds}.
\]

Exercise 3.
Let $M$ be a Riemannian manifold and let
\[
\Omega_{p,q} := \{c: [0, 1] \rightarrow M \mid c \text{ is piecewise smooth and } c(0) = p, c(1) = q\}.
\]
Show that a constant speed curve $c \in \Omega_{p,q}$ minimises the arc length functional $L: \Omega_{p,q} \rightarrow [0, \infty)$ if and only if it minimises the energy functional $E: \Omega_{p,q} \rightarrow [0, \infty)$.

Exercise 4.
Let $(M, g)$ be a complete simply-connected nonpositively curved Riemannian manifold and let $\lambda: \mathbb{R} \rightarrow M$ be a geodesic parametrised by arc length. For a point $p \in M \setminus \text{Im}(\lambda)$ define $d(s) = d_g(p, \lambda(s))$.

(a) Consider a family of geodesics $\gamma_s: [0, d(s)] \rightarrow M$ from $p$ to $\lambda(s)$ and show that
\[
\frac{1}{2} E'(s) = \langle \lambda'(s), \gamma'_s(d(s)) \rangle.
\]
(b) Conclude that $s_0$ is a critical point of $d$ if and only if $\langle \lambda'(s_0), \gamma'_{s_0}(d(s_0)) \rangle = 0$.

Exercise 5.
Let $M$ be a complete Riemannian manifold. Let $p, q$ be points in $M$ and let $\gamma: [0, a] \rightarrow M$ be a minimising geodesic joining $p$ to $q$. Show that, for all piecewise smooth curves $c: [0, a] \rightarrow M$ joining $p$ to $q$,
\[
E(\gamma) \leq E(c),
\]
with equality holding if and only if $c$ is a minimising geodesic.

Due: Wednesday May 6, 2015, before the exercise class.