Exercise 1.
Let \((M, g)\) be a Riemannian manifold and let \(c: [0, a] \to M\) be a piecewise smooth curve in \(M\) with constant speed and a variation \(f: (-\varepsilon, \varepsilon) \times [0, a] \to M\). Let \(L: (-\varepsilon, \varepsilon) \to \mathbb{R}\) be the arc length functional. Show that
\[
L'(0) = -\int_0^a \langle \frac{d}{dt} \frac{dc}{dt}, V(t) \rangle \, dt + \sum_{i=1}^{k-1} \langle V(t_i), \frac{dc}{dt}(t_i^+) - \frac{dc}{dt}(t_i^-) \rangle - \langle V(0), \frac{dc}{dt}(0) \rangle + \langle V(a), \frac{dc}{dt}(a) \rangle,
\]
where \(V(t)\) is the variational field of \(f\) and \(\frac{dc}{dt}(t_i^+) = \lim_{t \to t_i^+, t > t_i} \frac{dc}{dt}\) and \(\frac{dc}{dt}(t_i^-) = \lim_{t \to t_i^-, t < t_i} \frac{dc}{dt}\).

Exercise \(\star\) 2.
Show that a piecewise smooth curve \(c: [0, a] \to M\) is a geodesic if, and only if, for every proper variation \(f\) of \(c\) we have
\[
d \frac{d}{ds} E(0) = 0.
\]

Exercise 3.
Let \(\gamma: [0, a] \to M\) be an arbitrary geodesic and let \(f\) be an arbitrary variation of \(\gamma\). Show that in this case
\[
\frac{1}{2} E''(0) = -\int_0^a \langle V(t), \frac{D^2}{dt^2} V + R(\frac{d\gamma}{dt}, V, \frac{d\gamma}{dt}) \rangle \, dt - \sum_{i=1}^{k-1} \langle V(t_i), \frac{DV}{dt}(t_i^+) - \frac{DV}{dt}(t_i^-) \rangle - \langle V(0), \frac{DV}{dt}(0) \rangle + \langle V(a), \frac{DV}{dt}(a) \rangle.
\]

Exercise 4.
Let \(f: (-\varepsilon, \varepsilon) \times [0, a] \to M\) be a smooth variation of a smooth curve \(c: [0, a] \to M\). Show that
\[
\frac{D}{ds} \frac{D}{dt} \frac{\partial f}{\partial t} = \frac{D}{dt} \frac{D}{ds} \frac{\partial f}{\partial s} + R \left( \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right) \frac{\partial f}{\partial t},
\]
where \(R\) is the curvature tensor of \(M\).

Exercise 5.
Let \((M, g)\) be a simply-connected complete Riemannian manifold of nonpositive curvature and let \(\lambda: \mathbb{R} \to M\) be a geodesic parametrised by arc length. For a point \(p \in M \setminus \text{Im}(\lambda)\) define \(d(s) = d_g(p, \lambda(s))\). Show that
\[
\frac{1}{2} E''(s) > 0
\]
and conclude that \(d\) has exactly one critical point, which is a minimum.

Due: Wednesday May 13, 2015, before the exercise class.