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# Comparison Geometry in Summer 2015

## Exercise sheet 4

**Exercise 1.**
Let $(M, g)$ be a Riemannian manifold and let $c\colon [0, a] \to M$ be a piecewise smooth curve in $M$ with constant speed and a variation $f\colon (-\varepsilon, \varepsilon) \times [0, a] \to M$. Let $L\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$ be the arc length functional. Show that

$$L'(0) = -\int_0^a \left\langle \tfrac{\mathrm{D}}{\mathrm{d}\,t}\tfrac{\mathrm{d}\,c}{\mathrm{d}\,t}, V(t)\right\rangle \mathrm{d}\,t + \sum_{i=1}^{k-1} \left\langle V(t_i), \tfrac{\mathrm{d}\,c}{\mathrm{d}\,t}(t_i^+) - \tfrac{\mathrm{d}\,c}{\mathrm{d}\,t}(t_i^-)\right\rangle - \left\langle V(0), \tfrac{\mathrm{d}\,c}{\mathrm{d}\,t}(0)\right\rangle + \left\langle V(a), \tfrac{\mathrm{d}\,c}{\mathrm{d}\,t}(a)\right\rangle,$$

where $V(t)$ is the variational field of $f$ and $\frac{\mathrm{d}\,c}{\mathrm{d}\,t}(t^+) = \lim_{t\to t_i, t>t_i} \frac{\mathrm{d}\,c}{\mathrm{d}\,t}$ and $\frac{\mathrm{d}\,c}{\mathrm{d}\,t}(t^-) = \lim_{t\to t_i, t<t_i} \frac{\mathrm{d}\,c}{\mathrm{d}\,t}$.

**Exercise$^\star$ 2.**
Show that a piecewise smooth curve $c\colon [0, a] \to M$ is a geodesic if, and only if, for every proper variation $f$ of $c$ we have $\frac{\mathrm{d}}{\mathrm{d}\,s}E(0) = 0$.

**Exercise 3.**
Let $\gamma\colon [0, a] \to M$ be an arbitrary geodesic and let $f$ be an arbitrary variation of $\gamma$. Show that in this case

$$\begin{aligned}\tfrac{1}{2}E''(0) = &-\int_0^a \left\langle V(t), \frac{\mathrm{D}^2}{\mathrm{d}\,t^2}V + R\Big(\frac{\mathrm{d}\,\gamma}{\mathrm{d}\,t}, V\Big)\frac{\mathrm{d}\,\gamma}{\mathrm{d}\,t}\right\rangle \mathrm{d}\,t - \sum_{i=1}^{k-1} \left\langle V(t_i), \frac{\mathrm{D}\,V}{\mathrm{d}\,t}(t_i^+) - \frac{\mathrm{D}\,V}{\mathrm{d}\,t}(t_i^-)\right\rangle \\ &- \left\langle \frac{\mathrm{D}}{\mathrm{d}\,s}\frac{\partial f}{\partial s}, \frac{\mathrm{d}\,\gamma}{\mathrm{d}\,t}\right\rangle(0,0) + \left\langle \frac{\mathrm{D}}{\mathrm{d}\,s}\frac{\partial f}{\partial s}, \frac{\mathrm{d}\,\gamma}{\mathrm{d}\,t}\right\rangle(0,a) - \left\langle V(0), \frac{\mathrm{D}\,V}{\mathrm{d}\,t}(0)\right\rangle + \left\langle V(a), \frac{\mathrm{D}\,V}{\mathrm{d}\,t}(a)\right\rangle.\end{aligned}$$

**Exercise 4.**
Let $f\colon (-\varepsilon, \varepsilon) \times [0, a] \to M$ be a smooth varation of a smooth curve $c\colon [0, a] \to M$. Show that

$$\frac{\mathrm{D}}{\mathrm{d}\,s}\frac{\mathrm{D}}{\mathrm{d}\,t}\frac{\partial f}{\partial t} = \frac{\mathrm{D}}{\mathrm{d}\,t}\frac{\mathrm{D}}{\mathrm{d}\,s}\frac{\partial f}{\partial t} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t},$$

where $R$ is the curvature tensor of $M$.

**Exercise 5.**
Let $(M, g)$ be a simply-connected complete Riemannian manifold of nonpositive curvature and let $\lambda\colon \mathbb{R} \to M$ be a geodesic parametrised by arc lenght. For a point $p \in M \setminus \mathrm{Im}(\lambda)$ define $\mathrm{d}(s) = \mathrm{d}_g(p, \lambda(s))$. Show that

$$\frac{1}{2}E''(s) > 0$$

and conclude that d has exactly one critical point, which is a minimum.

Due: Wednesday May 13, 2015, before the exercise class.