

Comparison Geometry in Summer 2015 Exercise sheet 5

Let (M, g) be a Riemannian manifold and let $\gamma: [0, a] \rightarrow M$ be a geodesic with a variation $f: (-\varepsilon, \varepsilon) \times [0, a] \rightarrow M$. Let V be the variational field of f and let E be the energy function of the variation. Let

$$I_a(V, V) := \int_0^a \{ \langle V', V' \rangle - \langle R(\gamma', V)\gamma', V \rangle \} dt.$$

Exercise 1.

Show that if f is a proper variation, then $\frac{1}{2}E''(0) = I_a(V, V)$.

Exercise 2.

Show that if f is a general variation, then

$$\frac{1}{2}E''(0) = I_a(V, V) - \left\langle \frac{D}{ds} \frac{\partial f}{\partial s}, \gamma' \right\rangle(0, 0) + \left\langle \frac{D}{ds} \frac{\partial f}{\partial s}, \gamma' \right\rangle(0, a).$$