Exercise* 1 (Shorter curves in positive curvature).
Let \((M^{2n}, g)\) be an orientable and positively curved Riemannian manifold. Let \(\gamma\) be a closed geodesic in \(M\), i.e. \(\gamma: S^1 \to M\) is an immersion that is a geodesic at all of its points. Show that \(\gamma\) is homotopic to some closed curve \(c\) in \(M\) with \(L(c) < L(\gamma)\).

Exercise 2.
Show that the hypothesis \(\sec \geq \delta > 0\) in the corollary to the theorem of Bonnet-Myers is necessary. In other words, find a complete, non-compact Riemannian manifold whose sectional curvature is positive and not bounded away from 0.

Exercise 3.
Let \(M^n\) be a compact manifold with positive sectional curvature.

(i) Show that if \(n\) is even, orientability is necessary to conclude, via Synge’s Theorem, that \(M\) is simply connected.

(ii) Show that if \(n\) is odd, one cannot conclude that \(M\) is simply connected.

Exercise 4.
Show that completeness is necessary in the theorem of Bonnet-Myers to conclude that the fundamental group is finite.

Exercise 5.
Show that \(\mathbb{R}P^2 \times \mathbb{R}P^2\) has a metric with positive Ricci curvature but does not admit a metric with positive sectional curvature.

Due: Wednesday May 27th, 2015, before the exercise class.