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Comparison Geometry in Summer 2015

Exercise sheet 7

Exercise 1.
Let M be a Riemannian manifold and X a smooth vector field on M. Let $p \in M$ and let $U \subset M$ be a neighbourhood of p. Further, let $\phi\colon (-\varepsilon, \varepsilon) \times U \to M$ be a smooth mapping such that for any $q \in U$ the curve $t \mapsto \phi(t, q)$ is a trajectory of X passing through q at $t = 0$. (U and ϕ are given by the fundamental theorem for ordinary differential equations.) The field X is called a *Killing field* (or an *infinitesimal isometry*) if, for each $t_0 \in (-\varepsilon, \varepsilon)$, the mapping $\phi(t_0, \cdot)\colon M \supset U \to M$ is an isometry. Show that, if M is an even-dimensional, compact Riemannian manifold with positive sectional curvature, then every Killing field X of M vanishes at some point.

Exercise 2.
Let M be a compact Riemannian manifold with positive sectional curvature. Show that, if M is non-orientable, then every isometry of M has a fixed point.

Exercise* 3.
Let M^n be a complete, connected Riemannian manifold of dimension n with positive sectional curvature everywhere. Recall that a submanifold N of M is called *totally geodesic* if the geodesics with respect to the metric in N are geodesics in M. Show that any two compact, totally geodesic submanifolds A, B of M with $\dim A + \dim B \geq n$ must intersect.

Preparations. Let M be a Riemannian manifold and let $f\colon N \hookrightarrow M$ be an immersion of the manifold N. Every point $p \in N$ is contained in a sufficiently small neighbourhood $U \subset N$ such that $f(U)$ is an embedded submanifold in M. Thus, we will identify U and $f(U)$ in the following.

With respect to the metric g, the tangent bundle along U splits pointwise into a tangential and a normal part, i.e. $\mathrm{T}_p M = \mathrm{T}_p N \oplus \mathrm{T}_p N^\perp$. The Levi-Civita connection of the pullback metric on N can be given in terms of the Levi-Civita connection ∇^M in M via

$$\nabla_X Y = (\nabla^M_{\overline{X}} \overline{Y})^\top,$$

where $\overline{X}$ and $\overline{Y}$ are extensions of the local vector fields over M and $^\top$ denotes the projection to the tangential component. One can show that for a sufficiently small neighbourhood U in N

$$\mathcal{B}\colon \mathcal{V}(U) \times \mathcal{V}(U) \to \mathcal{V}(U)^\perp, \quad (X, Y) \mapsto \nabla^M_{\overline{X}} \overline{Y} - \nabla_X Y = \nabla^M_{\overline{X}} \overline{Y} - (\nabla^M_X Y)^\top$$

is a well-defined, bilinear and symmetric map, which only depends on the values of X and Y at the point p. The *second fundamental form* at $p \in N$ with respect to the immersion f and along the normal vector $\eta \in (\mathrm{T}_p N)^\perp$ is defined as the following quadratic form on $\mathrm{T}_p N$:

$$II_\eta(x) := g_p(\mathcal{B}(x, x), \eta).$$

Via $g_p(S_\eta(x), y) = g_p(B(x, y), \eta)$, the second fundamental form gives rise to a linear self-adjoint operator $S_\eta\colon \mathrm{T}_p N \to \mathrm{T}_p N$, which can be shown to be related to the connection in M via $S_\eta(x) = (\nabla^M_x N)^\top$.

Definition. The manifold N is called *minimal submanifold* of M if $\operatorname{tr} S_\eta(p) = 0$ for all $p \in N$ and $\eta \in (\mathrm{T}_p N)^\perp$.

Choosing an orthonormal frame field $E_1, \dots, E_k$ along U, where $k = \dim M - \dim N$, one can show that N is minimal if an only if

$$H(p) := \frac{1}{n} \sum_i (\operatorname{tr} S_{E_i}(p)) E_i(p) = 0$$

for all $p \in N$. The vector H does not depend on the frame $\{E_i\}$ and is called the *mean curvature vector.*

Exercise 4.
Let M^n be a complete, connected Riemannian manifold of positive Ricci curvature. Show that any two minimal hypersurfaces A, B, i.e. minimal submanifolds of dimension $n-1$, intersect in M.

You may proceed as follows:

(i) Choose an orthonormal frame field along a path connecting A and B together with corresponding variations and derive a total second variation of arc length.

(ii) Recognise a connection between the boundary terms and the mean curvature vectors to arrive at a contradiction using the curvature assumptions.

Due: Wednesday June 3rd, 2015, before the exercise class.