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Comparison Geometry in Summer 2015

Exercise sheet 8

Exercise 1.
Consider the gradient vector field $V = \nabla \operatorname{dist}(0, \cdot)$ on $\mathbb{R}^n$ and let $\gamma(t) = t\frac{\partial}{\partial x_n}$.

(i) Prove that $V = \sum_{i=1}^n \frac{x_i}{r}\frac{\partial}{\partial x_i}$ for $r^2 = \sum_{i=1}^n x_i^2$.

(ii) Let $A(t) = \nabla V|_{\gamma(t)}$. Prove that

$$A(t) = \frac{1}{t} I$$

with respect to the basis $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\}$.

Exercise* 2.
Let M_κ be a Riemannian manifold of constant curvature κ. Let $\gamma \colon \mathbb{R} \to M_\kappa$ be a geodesic and let $A(t) = a(t)I$ be a solution of the Riccati equation $A' + A^2 + R = 0$. Prove the following assertions:

(i) The function $a(t)$ is a solution of the equation $a' + a^2 + \kappa = 0$.

(ii) If $\kappa = 1$, then $a(t) = \cot(t - t_0)$ for some t_0.

(iii) If $\kappa = 0$, then $a(t) = \frac{1}{t - t_0}$ for some t_0 or $a(t) = 0$.

(iv) If $\kappa = -1$, then $a(t) = \coth(t - t_0)$ or $a(t) = \tanh(t - t_0)$ for some t_0, or $a(t) = \pm 1$.

Exercise 3.
Let $\gamma \colon \mathbb{R} \to M_\kappa$ be a geodesic and let $A(t)$ be a solution of the Riccati equation $A' + A^2 + R = 0$. Show that there exists a frame $E_1(t), \dots, E_{n-1}(t)$ of parallel vector fields along γ with respect to which $A(t)$ can be written as

$$A(t) = \operatorname{diag}(a_1(t), \dots, a_{n-1}(t)),$$

where the functions a_i are solutions of the equation $a_i' + a_i^2 + \kappa = 0$.

Exercise 4.
Let (M, g) be a Riemannian manifold and let $U \subset M$ be an open subset. A function $r \colon U \to \mathbb{R}$ is called a *distance function* if $|\nabla r| \equiv 1$ on U.

Show that $r \colon U \to [0, 1] \subset \mathbb{R}$ is a distance function if and only if r is a Riemannian submersion.

Due: Wednesday June 17th, 2015, before the exercise class.