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Comparison Geometry in Summer 2015
Exercise sheet 9

Exercise* 1.
Let M be a complete manifold with non-negative sectional curvature. Further, let $p_0, p_1 \in M$ be two points joined by a minimising geodesic $\gamma\colon [0,1] \to M$ and let X be a parallel vector field along γ such that $X \perp \gamma'$.

(1) Show that, for $p_s(t) = \exp tX(s)$ with $s \in [0,1]$,
$$|p_0(t), p_1(t)| \le |p_0, p_1|.$$

(2) Show that equality holds for some $t_0 > 0$ if and only if $p_0, p_1, p_0(t_0), p_1(t_0)$ bound a totally geodesic rectangle.

Exercise 2.
A smooth manifold M is said to have *almost non-positive curvature* if, for every $\varepsilon > 0$, there exists a complete Riemannian metric g_ϵ on M such that $\sec_{g_\varepsilon} \le K$ and $K\mathrm{diam}(M, g_\varepsilon)^2 \le \varepsilon$.

Show that S^2 does not have almost non-positive curvature.
Hint: Use the Gauß-Bonnet Theorem.

Exercise 3.
Give an alternative proof of the Bonnet-Myers theorem's weak form using Rauch I.

Theorem (Bonnet-Myers). Let (M, g) be a complete connected Riemannian manifold with $c \le \sec_g$ for some constant $c > 0$. Then M is compact and $\pi_1(M)$ finite.

Exercise 4.
Prove Klingenberg's long homotopy lemma.

Theorem (Klingenberg's Lemma). Let (M, g) be a complete Riemannian manifold with $\sec_g \le c$ for some constant $c > 0$. Further, let $p, q \in M$ be joined by two distinct, but homotopic, geodesics γ_0 and γ_1 with $L(\gamma_0) \le L(\gamma_1)$. Then there exists some $t_0 \in [0,1]$ such that
$$L(\gamma_0) + L(\alpha_{t_0}) \ge \frac{2\pi}{\sqrt{c}},$$
where α_t for $t \in [0,1]$ is the 1-parameter family of curves determined by a homotopy $H\colon \gamma_0 \simeq \gamma_1$.

You may want to proceed as follows:

(i) Use Rauch's theorem to show that $\exp_p$ has no critical points in the distance ball $\tilde{B} = \tilde{B}(0, \frac{\pi}{\sqrt{c}}) \subset \mathrm{T}_p M$.

(ii) Show that for small t, there exists a curve $\tilde{\alpha}_t$ in $\mathrm{T}_p M$ between $\exp_p^{-1}(p) = 0$ and $\exp_p^{-1}(q) = \tilde{q}$ such that $\exp_p \tilde{\alpha}_t = \alpha_t$.

(iii) Prove that α_1 cannot be lifted keeping the endpoints fixed.

(iv) By contradiction, show that for all $\varepsilon > 0$, there exists a $t(\varepsilon)$ such that $\alpha_{t(\varepsilon)}$ can be lifted to $\tilde{\alpha}_{t(\varepsilon)}$ and $\tilde{\alpha}_{t(\varepsilon)}$ contains points with distance $< \varepsilon$ from the boundary of B.

(v) Show that

$$L(\gamma_0) + L(\alpha_{t(\varepsilon)}) \geq \frac{2\pi}{\sqrt{c}} - 2\varepsilon$$

for all $\varepsilon > 0$ and conclude the inequality in Klingenberg's lemma.

Due: Wednesday July 1st, 2015, before the exercise class.