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Comparison Geometry in Summer 2015
Exercise sheet 10

Exercise* 1 (Rauch's theorem in dimension 2)**.**
Let

$$\begin{aligned} f''(t) + K(t)f(t) = 0, \quad f(0) = 0, \quad t \in [0,l], \\ \tilde{f}''(t) + \tilde{K}(t)\tilde{f}(t) = 0, \quad \tilde{f}(0) = 0, \quad t \in [0,l] \end{aligned}$$

be two differential equations with $\tilde{K}(t) \geq K(t)$ for $t \in [0,l]$ and $f'(0) = \tilde{f}'(0) = 1$.

(i) Show that for all $t \in [0,l]$ we have

$$0 = \int_0^t (\tilde{f}(f'' + Kf) - f(\tilde{f}'' + \tilde{K}\tilde{f})\,\mathrm{d}\,t = \Big[\tilde{f}f' - f\tilde{f}'\Big]_0^t + \int_0^t (K - \tilde{K})f\tilde{f}\,\mathrm{d}\,t.$$

Conclude that if $\tilde{f}(t) > 0$ on $(0, t_0)$ and $\tilde{f}(t_0) = 0$, then $f(t) > 0$ on $(0, t_0)$.

(ii) Let $\tilde{f}(t) > 0$ on $(0,l]$. Use part (i) and the fact that $f(t) > 0$ on $(0,l]$ to show that $f(t) \geq \tilde{f}(t)$ for all $t \in [0,l]$. Show that equality holds at $t = t_1 \in (0,l]$ if, and only if, $K(t) = \tilde{K}(t)$ for all $t \in [0,t_1]$.

(iii) Make sure that (ii) coincides with the theorem of Rauch in dimension 2.

Exercise 2 (Hyperbolic structure on a surface of genus $g \geq 2$.)**.**
Let $\mathbb{D}$ be 2-dimensional hyperbolic space (for example, the Poincaré model).

(1) Let $g \geq 2$. Show that there exists a regular hyperbolic $4g$-gon $P \subset \mathbb{D}$ with inner angles $\pi/2g$.

(2) Identify the edges of P pairwise following the labelling shown in Figure 1. Let Σ_g be the quotient space $P/\sim$ equipped with the quotient topology. Show that Σ_g is homeomorphic to a surface. (A neighborhood of the identification point is shown in Figure 2.)

(3) Let Γ be the set of curves γ in Σ_g that are continuous and piecewise smooth in the usual topology of $\mathbb{D}$ and are continous in the quotient topology. Let $L[\gamma]$ be the length of $\gamma \in \Gamma$ with respect to the hyperbolic metric. For $p, q \in \Sigma_g$, let Γ_{pq} be the set of curves in Γ from p to q. Define

$$d(p,q) = \inf\{\, L[\gamma] \mid \gamma \in \Gamma_{pq} \,\}.$$

Show that (Σ_g, d) is locally isometric to $\mathbb{D}$.

Due: Wednesday July 8th, 2015, before the exercise class.

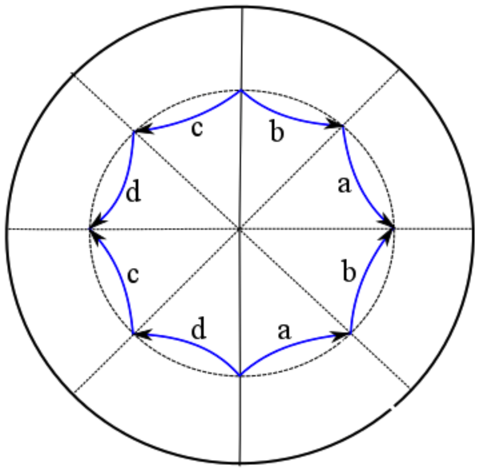

Figure 1: Identification scheme

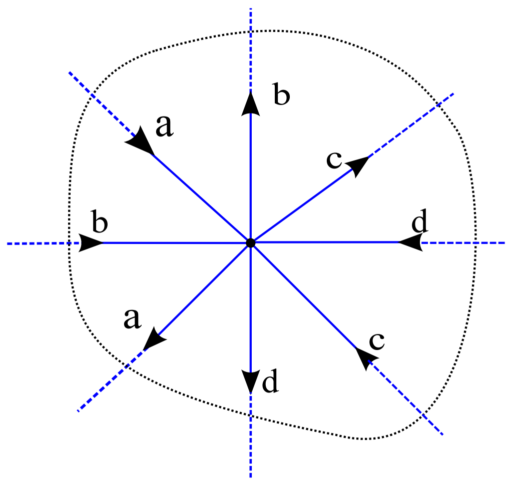

Figure 2: Neighborhood of the identification point