Exercise 1 (Rauch’s theorem in dimension 2).

Let
\[ f''(t) + K(t)f(t) = 0, \quad f(0) = 0, \quad t \in [0, l], \]
\[ \tilde{f}''(t) + \tilde{K}(t)\tilde{f}(t) = 0, \quad \tilde{f}(0) = 0, \quad t \in [0, l] \]
be two differential equations with \( \tilde{K}(t) \geq K(t) \) for \( t \in [0, l] \) and \( f'(0) = \tilde{f}'(0) = 1 \).

(i) Show that for all \( t \in [0, l] \) we have
\[ 0 = \int_0^t (\tilde{f}(f'' + Kf) - f(\tilde{f}'' + \tilde{K}\tilde{f})) \, dt = \left[ \tilde{f}f' - f\tilde{f}' \right]_0^t + \int_0^t (K - \tilde{K})f\tilde{f} \, dt. \]

Conclude that if \( \tilde{f}(t) > 0 \) on \( (0, t_0) \) and \( \tilde{f}(t_0) = 0 \), then \( f(t) > 0 \) on \( (0, t_0) \).

(ii) Let \( \tilde{f}(t) > 0 \) on \( (0, l] \). Use part (i) and the fact that \( f(t) > 0 \) on \( (0, l] \) to show that \( f(t) \geq \tilde{f}(t) \) for all \( t \in [0, l] \). Show that equality holds at \( t = t_1 \in (0, l] \) if, and only if, \( K(t) = \tilde{K}(t) \) for all \( t \in [0, t_1] \).

(iii) Make sure that (ii) coincides with the theorem of Rauch in dimension 2.

Exercise 2 (Hyperbolic structure on a surface of genus \( g \geq 2 \)).

Let \( D \) be 2-dimensional hyperbolic space (for example, the Poincaré model).

(1) Let \( g \geq 2 \). Show that there exists a regular hyperbolic \( 4g \)-gon \( P \subset D \) with inner angles \( \pi/2g \).

(2) Identify the edges of \( P \) pairwise following the labelling shown in Figure 1. Let \( \Sigma_g \) be the quotient space \( P/\sim \) equipped with the quotient topology. Show that \( \Sigma_g \) is homeomorphic to a surface. (A neighborhood of the identification point is shown in Figure 2.)

(3) Let \( \Gamma \) be the set of curves \( \gamma \) in \( \Sigma_g \) that are continuous and piecewise smooth in the usual topology of \( D \) and are continuous in the quotient topology. Let \( L[\gamma] \) be the length of \( \gamma \in \Gamma \) with respect to the hyperbolic metric. For \( p, q \in \Sigma_g \), let \( \Gamma_{pq} \) be the set of curves in \( \Gamma \) from \( p \) to \( q \). Define
\[ d(p, q) = \inf \{ L[\gamma] \mid \gamma \in \Gamma_{pq} \}. \]

Show that \( (\Sigma_g, d) \) is locally isometric to \( D \).

Due: Wednesday July 8th, 2015, before the exercise class.
Figure 1: Identification scheme

Figure 2: Neighborhood of the identification point