Karlsruher Institut für Technologie

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# Comparison Geometry in Summer 2015

## Exercise sheet 11

**Exercise 1** (Existence of a small basis)**.**
Let $(M, g)$ be a compact connected Riemannian manifold. Show that its fundamental group $\pi := \pi_1(M, p_0)$ is finitely generated.

Certainly, $\pi$ is discrete and acts freely by isometries on the universal cover $\tilde{M}$.

(i) Consider the so-called *Dirichlet Fundamental domain* for a fixed point $x_0 \in \tilde{M}$

$$F := \{x \in \tilde{M} \mid \ |x, x_0| \leq |x, l \cdot x_0| \ \forall l \in \pi\}.$$

Show that $F \subset \tilde{M}$ is a closed subset with $\operatorname{int}(F) \cap \operatorname{int}(l \cdot F) = \emptyset$ for all $l \in \pi$, $l \neq 1$, and $\pi \cdot F = \tilde{M}$.

(ii) Call an element $l \in \pi$ *small* if $F \cap l \cdot F \neq \emptyset$. Show that the set of all small elements is a finite generating set of $\pi$.

**Exercise 2** (Fundamental group of manifolds with non-negative curvature)**.**
Prove the following theorem.

**Theorem (Gromov, 1978).** Let $(M^n, g)$ be a complete connected Riemannian manifold with $\sec_g \geq 0$. Then $\pi := \pi_1(M, p_0)$ contains a generating set of $N \leq C(n)$ elements, where $C(n)$ is a number only depending on the dimension $n$ of $M$.

You may want to proceed as follows.

Define $|l| := |p, l \cdot p|$ for all $l \in \pi$ and pick $l_1$ with $|l_1|$ minimal. Inductively define

$$\Gamma_k := \langle l_1, \ldots, l_k \rangle \subset \pi$$

with $l_{k+1} \in \pi \setminus \Gamma_k$ with $|l_{k+1}|$ minimal. Now it suffices to show that $\Gamma_k = \pi$ for some $k \leq C(n) := 2\sqrt{5}^n$.

(i) Consider the triangle $p, p_i := g_i \cdot p, p_j := g_j \cdot p$ for $j > i$ and use Toponogov's theorem to conclude that $\alpha_{ij} \geq 60°$ for all $i \neq j$, where $\alpha_{ij}$ is the angle between $v_i$ and $v_j$ such that $\gamma_{v_i} \colon p \rightsquigarrow p_i$ and $\gamma_{v_j} \colon p \rightsquigarrow p_j$ are shortest geodesics joining, respectively, $p$ to $p_i$ and $p$ to $p_j$.

(ii) Notice that for any two such vectors $v_i, v_j$, the balls with radius $\frac{1}{2}$ centered at $v_i$ and $v_j$ are disjoint and their inner half balls are contained in $B_{\frac{\sqrt{5}}{2}}(0)$. Deduce, by comparing the volume of $B_{\frac{\sqrt{5}}{2}}(0)$ with the volume of $B_{\frac{1}{2}}$, that there can be at most $2\sqrt{5}^n$ vectors such that $\alpha_{ij} \geq 60°$.

Due: Wednesday July 15th, 2015, before the exercise class.