Institut für Algebra und Geometrie

Prof. Dr. W. Tuschmann
Dipl.-Math. M. Herrmann

Karlsruher Institut für Technologie

Global Differential Geometry
Exercise sheet 1

Exercise 1

Let (X, d) be a metric space and let $\mathcal{C}(X)$ denote the set of all closed, bounded and non-empty subsets of X. Let d_H^X denote the Hausdorff distance on X. Show that $(\mathcal{C}(X), d_H^X)$ is in fact a metric space.

Exercise 2

Let (X, d) be a compact metric space and $(A_i)_{i\in\mathbb{N}}$ a sequence in $\mathcal{C}(X)$. Show that

(a) If $A_{i+1} \subset A_i$ for all i, then (A_i) converges to $\bigcap_{i\in\mathbb{N}} A_i$ with respect to the Hausdorff distance.

(b) If $A_i \subset A_{i+1}$ for all i, then (A_i) converges to the closure of $\bigcup_{i\in\mathbb{N}} A_i$ with respect to the Hausdorff distance.

Find a counterexample to (b) for X noncompact.

Exercise 3

For a map $f : X \to Y$between metric spaces X and Y define the *distortion* $\operatorname{dis} f$ of f by

$$\operatorname{dis} f = \sup_{x_1,x_2\in X} |d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)|.$$

Let $\varepsilon > 0$ and f be an ε isometry, i.e. $\operatorname{dis} f \leq \varepsilon$ and $f(X)$ is an ε-net in Y.

Show that $d_{GH}(X, Y) < 2\varepsilon$.

Exercise 4

Let $X_n := \{(\frac{k}{n}, y), (x, \frac{k}{n}) \mid 0 \leq k \leq n, x, y \in [0, 1]\}$ be a lattice in the unit square. A metric is given by

$$d_{X_n}(x, y) = \text{ Length of a shortest path in } X_n \text{ connecting } x \text{ and } y,$$

where the length of a path is the length in $\mathbb{R}^2$ with the usual metric. Show that (X_n) converges to $[0, 1]^2$ with the "Manhattan" metric, i.e. $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$ for $x, y \in [0, 1]^2$.

Due: Friday, 4.5.2012, during the exercise class