Exercise 1 (Berger Spheres)
We can identify $S^3 \subset \mathbb{C}^2$ with $SU(2)$ by

$$S^3 \ni (z, w) \mapsto \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \in SU(2).$$

We consider the basis

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

of $su(2)$ and the left invariant metric $g_\varepsilon$, given by $g_\varepsilon(X_1, X_1) = \varepsilon$, $g_\varepsilon(X_2, X_2) = g_\varepsilon(X_3, X_3) = 1$ and $g_\varepsilon(X_i, X_j) = 0$ for $i \neq j$.

Show that $(S^3, g_\varepsilon) \to S^2(r)$ for an $r > 0$ in the Gromov-Hausdorff topology, where $S^2(r)$ is the round sphere of radius $r$.

Exercise 2
Show that the class of Riemannian $(\leq n)$-manifolds with $\sec \geq \kappa$ and $\diam \leq D$ is (in general) not compact in the Gromov-Hausdorff topology.

Exercise 3
Let $M$ be a compact smooth manifold and $f$ a diffeomorphism of $M$. Let $g_1$ and $g_2$ be Riemannian metrics on $M$. We regard $f$ as a map $f : (M, g_1) \to (M, g_2)$. Let $\nu_1 = \sup_{p \in M} \|f_\ast p\|$ and $\nu_2 = \sup_{p \in M} \|f^{-1}_\ast p\|$, where $\|\cdot\|$ is the operator norm.

Show that for $\varepsilon = \max\{(\nu_1 - 1)\diam(M, g_1), (1 - 1/\nu_2)\diam(M, g_1)\}$, $f$ is an $\varepsilon$-isometry.