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Global Differential Geometry
Exercise sheet 2

Exercise 1 (Berger Spheres)
We can identify $\mathrm{S}^3 \subset \mathbb{C}^2$ with $\mathrm{SU}(2)$ by

$$\mathrm{S}^3 \ni (z, w) \mapsto \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \in \mathrm{SU}(2).$$

We consider the basis

$$X_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

of $\mathfrak{su}(2)$ and the left invariant metric g_ε, given by $g_\varepsilon(X_1, X_1) = \varepsilon$, $g_\varepsilon(X_2, X_2) = g_\varepsilon(X_3, X_3) = 1$ and $g_\varepsilon(X_i, X_j) = 0$ for $i \neq j$.

Show that $(\mathrm{S}^3, g_\varepsilon) \to \mathrm{S}^2(r)$ for an $r > 0$ in the Gromov-Hausdorff topology, where $\mathrm{S}^2(r)$ is the round sphere of radius r.

Exercise 2
Show that the class of Riemannian $(\leq n)$-manifolds with $\sec \geq \kappa$ and $\operatorname{diam} \leq D$ is (in general) not compact in the Gromov-Hausdorff topology.

Exercise 3
Let M be a compact smooth manifold and f a diffeomorphism of M. Let g_1 and g_2 be Riemannian metrics on M. We regard f as a map $f : (M, g_1) \to (M, g_2)$. Let $\nu_1 = \sup_{p \in M} \|f_{*_p}\|$ and $\nu_2 = \sup_{p \in M} \|f_{*_p}^{-1}\|$, where $\|.\|$ is the operator norm.

Show that for $\varepsilon = \max\{(\nu_1 - 1)\operatorname{diam}(M, g_1), (1 - 1/\nu_2)\operatorname{diam}(M, g_1)\}$, f ist an ε-isometry.

Due: Friday, 11.5.2012, during the exercise class