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Global Differential Geometry
Exercise sheet 4

Exercise 1

(a) Show that in the Hopf-Rinow theorem the local compactness is necessary, i.e. find a complete length space X in which some points cannot be connected by a shortest geodesic.

(b) Show that the completion of a length space is again a length space. Is the completion of a locally compact length space locally compact?

Exercise 2 (Products and Cones)

For metric spaces (X, d_X) and (Y, d_Y) define the *direct product* as the metric space $X \times Y$ with the metric $d((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$.

Show that if d_X and d_Y are (strictly) intrinsic, the direct product metric is (strictly) intrinsic, too.

For a metric space (X, d) with $\operatorname{diam}(X, d) \leq \pi$ we define $\operatorname{Con}(X)$, the *cone over* X, by $\operatorname{Con}(X) = X \times [0, \infty)/X \times \{0\}$ with the metric

$$d_c(p, q) = \sqrt{t^2 + s^2 - 2ts\cos(d(x, y))},$$

where $p = [(x, t)]$ and $q = [(y, s)]$. Then d_c is a metric and if d is intrinsic, so is d_c.

Show that for $X = \mathrm{S}^n$ with the metric induced by its standard Riemannian metric, $\operatorname{Con}(X)$ is isometric to $\mathbb{R}^{n+1}$ with the euclidean metric.

Exercise 3

Show that the metric spaces X_n of exercise 1.4 do not have a lower curvature bound.

Let d be the metric on $\mathbb{R}^n$ induced by the 1-norm $\|.\|_1$ (so $d(x, y) = \|x - y\|_1 = \sum |x_i - y_i|$). Show that $(\mathbb{R}^n, d)$ has no lower curvature bound.

Due: Friday, 25.5.2012, during the exercise class