Karlsruher Institut für Technologie

Institut für Algebra und Geometrie

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Global Differential Geometry
Exercise sheet 5

Exercise 1

Prove the Weitzenböck formula: Let (M, g) be a Riemannian manifold and $f \in \mathrm{C}^3(M)$, then

$$\tfrac{1}{2}\Delta(|\nabla f|^2) = |\mathrm{Hess}(f)|^2 + \langle \nabla f, \nabla(\Delta f)\rangle + \mathrm{Ric}(\nabla f, \nabla f).$$

Hint: To prove the equality in a point $p \in M$ use local orthonormal vector fields X_i, s.t. $\nabla_{X_i} X_j(p) = 0$.

Exercise 2

Let M be a noncompact complete Riemannian manifold of nonnegative Ricci curvature and $p \in M$. Show that there exists $C > 0$ such that $\mathrm{vol}B_p(r) \geq Cr$ for all large enough r.

Hint: In a noncompact complete Riemannian manifold, there exists a ray (i.e. a geodesic $\gamma : [0, \infty) \to M$ s.t. $d(\gamma(s), \gamma(t)) = |s - t|$ for all s and t) (why?) and from what we proved in the lecture, it follows that we have a version of the relative volume comparison, where the ball of larger radius is replaced by an annullus with outer radius larger than the radius of the other ball.

Exercise 3

Let G be a finitely generated abelian group. What is the growth rate of G?

Due: Friday, 1.6.2012, during the exercise class