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Global Differential Geometry
Exercise sheet 7

Exercise 1

Let G be a Lie group and $\langle .\,,. \rangle$ a left-invariant metric on G. Show that for left-invariant vector fields X and Y

$$\nabla_X Y = \frac{1}{2}\big([X,Y] - (\mathrm{ad}_X)^*(Y) - (\mathrm{ad}_Y)^*(X)\big),$$

where $\mathrm{ad}_X(Y) = [X,Y]$ and * denotes the adjoint map with respect to $\langle .\,,. \rangle$.

Exercise 2

Let

$$G := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}.$$

G is called the *Heisenberg group*.

Show that there is a basis ξ_1, ξ_2, ξ_3 of the Lie algbra $\mathfrak{g}$ of G, such that $[\xi_1, \xi_2] = \xi_3$ and $[\xi_1, \xi_3] = [\xi_2, \xi_3] = 0$.

Exercise 3

Let G, $\mathfrak{g}$ and ξ_i, $i = 1, 2, 3$, as above. For $0 < q \leq 1$, we define a left-invariant metric g_q on G by

$$g_q(\xi_1, \xi_1) = q^2 = g_q(\xi_2, \xi_2), \quad g_q(\xi_3, \xi_3) = q^4 \;\text{ and }\; g_q(\xi_i, \xi_j) = 0 \text{ for } i \neq j.$$

Show that $-\frac{3}{4} \leq \sec(M, g_q) \leq \frac{1}{4}$ for all $q \in (0, 1]$.

Due: Thursday, 21.7.2012, during the exercise class