Institut für Algebra und Geometrie
Prof. Dr. W. Tuschmann
Dipl.-Math. M. Herrmann

Global Differential Geometry
Exercise sheet 9

Exercise 1

Prove the π_1-finiteness theorem in dimension 4. **Hint:** Use the Poincaré-Hopf theorem.

Exercise 2

For $k, l \in \mathbb{Z}$, $|k| + |l| \neq 0$, let

$$T_{k,l} = \left\{ \mathrm{diag}(e^{ikt}, e^{ilt}, e^{-i(k+l)t}) \middle| t \in \mathbb{R} \right\} \subset \mathrm{SU}(3)$$

and $W_{k,l}$ the Aloff-Wallach space $W_{k,l} = \mathrm{SU}(3)/T_{k,l}$. Assume $kl > 0$.

We equip the Lie algebra $\mathfrak{su}(3)$ of $\mathrm{SU}(3)$ with the biinvariant metric $\langle X, Y\rangle_0 = -\mathrm{Re}(\mathrm{tr}(XY))$.

Show that there exist nontrivial linear subspaces V_1 and V_2 of $\mathfrak{su}(3)$ such that

- the orthogonal complement $\mathfrak{p}$ of the Lie algebra $\mathfrak{t}_{k,l}$ of $T_{k,l}$ splits as the orthogonal direct sum $\mathfrak{p} = V_1 \oplus V_2$,
- $[V_1, V_2] \subset V_2$, $\quad [V_1, V_1] \subset V_1 + \mathfrak{t}_{k,l}$, $\quad [V_2, V_2] \subset V_1 + \mathfrak{t}_{k,l}$, and
- $\mathrm{Ad}(T_{k,l})V_i \subset V_i$, for $i = 1, 2$.
- For all $x = x_1 + x_2, y = y_1 + y_2 \in V_1 \oplus V_2$: If $[x, y] = 0$ and $x \wedge y \neq 0$ (i.e. $\mathrm{span}\{x, y\}$ is two-dimensional), then $[x_1, y_1] \neq 0$.

Exercise 3 (*)

Let g_t denote the metric on $M = \mathrm{SU}(3)/T_{k,l}$ induced by $\langle X, Y\rangle_t = (1+t)\langle X_1, Y_1\rangle_0 + \langle X_2, Y_2\rangle_0$, where the subscripts denote the projections on V_1 and V_2. Show that (M, g_t) has strictly positive sectional curvature for $-1 < t < 0$.