Exercise 1
Let $M^n$ and $N^{n+k}$ be smooth manifolds of dimension $n$ and $n+k$, respectively. Recall that a smooth map $f : M^n \to N^{n+k}$ is an immersion if $df_p : T_p M \to T_{f(p)} N$ is injective for all $p$ in $M$. If $N$ has a Riemannian metric, show that $f$ induces a Riemannian metric on $M$ by defining, for all $p \in M$,

$$\langle u, v \rangle_p := \langle df_p u, df_p v \rangle_{f(p)}, \quad u, v \in T_p M.$$ 

This metric on $M$ is called the metric induced by $f$, and $f$ is said to be an isometric immersion.

Exercise 2
There are two kinds of “metrics” on a Riemannian manifold: the Riemannian metric and the distance function induced by the length of piecewise smooth curves. Correspondingly, there are two definitions of “isometry” between Riemannian manifolds: a Riemannian isometry is a diffeomorphism that pulls one Riemannian metric back to the other, and a metric isometry is a homeomorphism that pulls one distance function back to the other. Prove that these two kinds of isometry are identical.

Hint: For the hard direction, first use the exponential map to show the homeomorphism is smooth.

Exercise 3
Let $M$ and $N$ be Riemannian manifolds. A smooth map $f : M \to N$ is a local isometry at $p \in M$ if there is a neighborhood $U \subset M$ of $p$ such that the map $f : U \to f(U)$ is a diffeomorphism satisfying

$$\langle u, v \rangle_p = \langle df_p u, df_p v \rangle_{f(p)}, \quad \text{for all } p \in M, \ u, v \in T_p M$$

(i.e. the diffeomorphism $f : U \to f(U)$ is an isometry).

Recall that the set

$$S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}$$

is the unit sphere of $\mathbb{R}^{n+1}$ and the metric induced on $S^n$ from $\mathbb{R}^{n+1}$ (with its usual Euclidean metric) is called the canonical metric of $S^n$.

Show that the antipodal map $A : S^n \to S^n$ given by $A(x) = -x$ is an isometry of $S^n$. Use this fact to introduce a Riemannian metric on the real projective space $\mathbb{R} P^n$ such that the natural projection $\pi : S^n \to \mathbb{R} P^n$ is a local isometry.

Due: Friday, 2.5.2014, during the exercise class.