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Global Differential Geometry
Exercise sheet 1

Exercise 1

Let M^n and N^{n+k} be smooth manifolds of dimension n and $n+k$, respectively. Recall that a smooth map $f : M^n \to N^{n+k}$ is an *immersion* if $df_p : T_pM \to T_{f(p)}N$ is injective for all p in M. If N has a Riemannian metric, show that f induces a Riemannian metric on M by defining, for all $p \in M$,

$$\langle u, v\rangle_p := \langle df_p u, df_p v\rangle_{f(p)}, \quad u, v \in T_pM.$$

This metric on M is called the metric *induced* by f, and f is said to be an *isometric immersion*.

Exercise 2

There are two kinds of "metrics" on a Riemannian manifold: the Riemannian metric and the distance function induced by the length of piecewise smooth curves. Correspondingly, there are two definitions of "isometry" between Riemannian manifolds: a *Riemannian isometry* is a diffeomorphism that pulls one Riemannian metric back to the other, and a *metric isometry* is a homeomorphism that pulls one distance function back to the other. Prove that these two kinds of isometry are identical.

Hint: For the hard direction, first use the exponential map to show the homeomorphism is smooth.

Exercise 3

Let M and N be Riemannian manifolds. A smooth map $f : M \to N$ is a *local isometry* at $p \in M$ if there is a neighborhood $U \subset M$ of p such that the map $f : U \to f(U)$ is a diffeomorphism satisfying

$$\langle u, v\rangle_p = \langle df_p u, df_p v\rangle_{f(p)}, \quad \text{for all } p \in M,\ u, v \in T_pM$$

(i.e. the diffeomorphism $f : U \to f(U)$ is an isometry).

Recall that the set

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$$

is the *unit sphere* of $\mathbb{R}^{n+1}$ and the metric induced on $\mathbb{S}^n$ from $\mathbb{R}^{n+1}$ (with its usual Euclidean metric) is called the *canonical metric* of $\mathbb{S}^n$.

Show that the antipodal map $A : \mathbb{S}^n \to \mathbb{S}^n$ given by $A(x) = -x$ is an isometry of $\mathbb{S}^n$. Use this fact to introduce a Riemannian metric on the real projective space $\mathbb{R}P^n$ such that the natural projection $\pi : \mathbb{S}^n \to \mathbb{R}P^n$ is a local isometry.

Due: Friday, 2.5.2014, during the exercise class.