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Global Differential Geometry
Exercise sheet 2

Exercise 1

Consider $\mathbb{R}^n$ equipped with the Euclidean Riemannian metric.

(a) Let $\mathrm{E}(n)$ be the set of $(n+1)\times(n+1)$ real matrices of the form $\begin{pmatrix} U & v \\ 0 & 1 \end{pmatrix}$, where $U \in \mathrm{O}(n)$ and $v \in \mathbb{R}^n$ (considered as a column vector). Prove that $\mathrm{E}(n)$ is a closed Lie subgroup of $\mathrm{GL}(n+1,\mathbb{R})$. We call $\mathrm{E}(n)$ the *Euclidean group* or the *group of rigid motions*.

(b) Define a map $\mathrm{E}(n) \times \mathbb{R}^n \to \mathbb{R}^n$ by identifying $\mathbb{R}^n$ with the set

$$S = \{(x,1) \in \mathbb{R}^{n+1} \,:\, x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$$

and restricting the linear action of $\mathrm{E}(n)$ on $\mathbb{R}^{n+1}$ to S. Prove that this map is a smooth action of $\mathrm{E}(n)$ on $\mathbb{R}^n$ by isometries of the Euclidean metric.

(c) Show that $\mathrm{E}(n)$ acts transitively on $\mathbb{R}^n$ and takes any orthonormal basis to any other one.

Exercise 2

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A Riemannian metric g on G is said to be *left-invariant* if it is invariant under all left translations, i.e. $L_x^* g = g$ for all $x \in G$. Similarly, g is *right-invariant* if it is invariant under all right translations and *bi-invariant* if it is both left- and right-invariant.

(a) Show that a metric g is left-invariant if and only if the coefficients $g_{ij} = g(X_i, X_j)$ with respect to any left-invariant frame $\{X_i\}$ are constants.

(b) Show that the restriction map $g \mapsto g|_{T_eG}$ gives a bijection between left-invariant metrics on G and inner products on $\mathfrak{g}$.

Exercise 3

Let G be a compact, connected Lie group with a left-invariant metric g, and let dV be the Riemannian volume element of g. Show that dV is bi-invariant.
Hint: Show that $R_x^* dV$ is left-invariant and positively oriented, and is therefore equal to $\varphi(x)dV$ for some positive number $\varphi(x)$. Show that $\varphi : G \to \mathbb{R}^+$ is a Lie group homomorphism, so its image is a compact subgroup of $\mathbb{R}^+$.

Exercise 4

Let G be a Lie group and fix $x \in G$. Conjugation by x induces a Lie group automorphism $C_x : G \to G$, given by $y \mapsto xyx^{-1}$, $y \in G$ and called an *inner automorphism*. Let $\mathrm{Ad}_x = (C_x)_* : \mathfrak{g} \to \mathfrak{g}$ be the induced Lie algebra automorphism.

(a) Verify that $C_{x_1} \circ C_{p_2} = C_{x_1 x_2}$, so that $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$ is a representation of G, called the *adjoint representation*.

(b) Show that an inner product on $\mathfrak{g}$ induces a bi-invariant metric on G if and only if the inner product is invariant under the adjoint representation.

Due: Friday, 9.5.2014, during the exercise class.