Karlsruher Institut für Technologie

Institut für Algebra und Geometrie

Prof. Dr. W. Tuschmann
Dr. F. Galaz-García

Global Differential Geometry

Exercise sheet 3

Exercise 1

Let M be a compact Riemannian manifold with an isometric action of a compact Lie group G.

(a) Show that each orbit of the action is a closed subset of M.

(b) Let p, q be points in M with orbits $G(p)$, $G(q)$, respectively. Show that

$$d(G(p), G(q)) = \inf\{d(x, y) : x \in G(p),\ y \in G(q)\}$$

is a metric on the orbit space.

Exercise 2

Let

$$G = \left\{ \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \middle| x, y, z \in \mathbb{R} \right\}.$$

(a) Show that G is a Lie group. This group is called the *Heisenberg group*.

(b) Show that there is a basis ξ_1, ξ_2, ξ_3 of the Lie algebra $\mathfrak{g}$ of G, such that $[\xi_1, \xi_2] = \xi_3$ and $[\xi_1, \xi_3] = [\xi_2, \xi_3] = 0$.

Exercise 3

Show that every compact, connected Lie group admits a bi-invariant Riemannian metric.
Hint: Start with an arbitrary inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ and integrate the function f defined by $f(p) = \langle \mathrm{Ad}_p X, \mathrm{Ad}_p Y \rangle$ over the group. You will need to use the result of Exercise 3 in Sheet 2.

Exercise 4

Show that $\mathfrak{so}(n) = \{A \in M_n(\mathbb{R}) \mid A^T = -A\}$.

Due: Friday, 9.16.2014, during the exercise class.