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Global Differential Geometry
Exercise sheet 7

Exercise 1

Consider the group $G = \{\pm 1\}$ acting on $\mathbb{R}$ by multiplication. The circle $B = \mathbb{S}^1$ of unit complex numbers has an open cover given by the sets $U_1 = \mathbb{S}^1 \setminus \{-i\}$ and $U_2 = \mathbb{S}^1 \setminus \{i\}$. Show that the map $f_{1,2} : U_1 \cap U_2 \to G$ given by

$$z \quad \mapsto \quad \begin{cases} 1, & \text{if } \mathrm{Re} z > 0, \\ -1, & \text{otherwise,} \end{cases}$$

determines a rank 1 vector bundle over the circle, called a *Möbius band*.

Exercise 2

Let $\xi = (E, \pi, B, F)$ be a fiber bundle. A map $s : B \to M$ is said to be a *cross-section* of ξ if $\pi \circ s = \mathrm{Id}_B$. Show that a principal G-bundle $\pi : P \to B$ admits a section if and only if it is trivial.

Exercise 3

Let G be a compact connected Lie group and let $M = G/H$ be a homogeneous space. The *linear isotropy representation* at $p = eH \in M$ is the homomorphism $\rho : H \to \mathrm{GL}(T_pM)$ given by $\rho(h) = h_{*p}$.

(a) Show that, if G acts effectively on M, then ρ is one-to-one and induces an effective linear action of H on T_pM.

(b) Suppose that G acts effectively on M. Show that the tangent bundle of M is equivalent to the bundle $G \times_H T_pM \to M$, where H acts on T_pM via the linear isotropy representation at p. Hint: Consider the map $f : G \times_H T_pM \to TM$ defined by $f[g, u] = g_* u$.

Exercise 4

Let $M = G/H$ be a homogeneous space.

(a) Show that the subgroup of G which acts trivially on M is the largest normal subgroup $N(H)$ of G which lies in H.

(b) Show that $\overline{G} = G/N(H)$ acts effectively on M, and that $M = \overline{G}/\overline{H}$, where $\overline{H} = H/N(H)$.

It follows from this exercise that the hypothesis that G act effectively in Exercise 3 is not restrictive.

Due: Friday, 20.6.2014, during the exercise class.