Global Differential Geometry

Exercise sheet 9

Exercise 1
Let $M$ be a smooth manifold and let $\mathcal{X}(M)$ be the set of smooth vector fields on $M$. Show that the Lie derivative $\mathcal{L} : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ is not a connection.

Exercise 2
Let $M$ be a smooth manifold with a linear connection $\nabla$. Show that the map

$$\mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)
\quad (X, Y, Z) \mapsto \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$$

defines a $(3, 1)$ tensor field.

Exercise 3
A Riemannian manifold $M$ is isotropic at $p \in M$ if there exists a Lie group $G$ acting by isometries on $M$ such that the isotropy group $G_p \leq G$ acts transitively on the unit tangent sphere at $p$. We say that $M$ is isotropic if it is isotropic at every point. Suppose that $M$ is a complete Riemannian manifold that is isotropic. Show that $M$ is homogeneous.

Hint: Given $p, q \in M$, consider the midpoint of a geodesic in $M$.

Exercise 4
Show that a homogeneous Riemannian manifold is complete.

Due: Friday, 4.7.2014, during the exercise class.