Moduli Spaces of Metrics with Lower or Pinched Curvature Bounds and their Compactifications

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Dedicated to Xiaochun Rong on the occasion of his 70th birthday

Abstract We present a metric approach to the study of moduli spaces of metrics with certain curvature bounds and suitable other geometric constraints, and their compactifications. This is accompanied by a deeper discussion of related results, questions and problems in the realm of positive and positively pinched sectional as well as Ricci curvature. Regarding the latter two topics, we place special emphasis on corresponding works and contributions of X. Rong and his collaborators.

1 Introduction

A central theme of global Riemannian geometry consists in the construction of Riemannian metrics with certain geometric constraints like, say, curvature bounds, on a given smooth manifold M. However, once the existence problem has been solved, an equally important task is then to study the question what the space of all such metrics on M does look like. Moreover, one also seeks to understand the structure of the corresponding moduli space of these metrics. The latter is defined as the quotient of the former by the diffeomorphism group of the manifold, acting on the space of metrics or suitable subspaces by pulling back metrics.

These spaces are customarily equipped with the topology of smooth convergence on compact subsets and the corresponding quotient topology, respectively. Their topological properties then provide the right means to measure 'how many' different metrics, or geometries, respectively, the given manifold actually does exhibit. For further details, compare the monograph [31].

In the present note, we first propose and use a different method to study moduli spaces of Riemannian metrics. For simplicity, let us assume that the manifold M in question is compact. We can then view a moduli space $\mathcal{M}(M)$ of metrics on M in the

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former sense also as a subspace of the Gromov-Hausdorff space $(\mathcal{M}, d_{GH})$ of isometry classes of compact metric spaces, equipped with the Gromov-Hausdorff distance. Indeed, since smooth (or C^k or $C^{k,\alpha}$) convergence implies Gromov-Hausdorff convergence, one can think of $\mathcal{M}(M)$ as being continuously embedded into $(\mathcal{M}, d_{GH})$. [1] Moreover, this point of view also allows to obtain and study natural subspace compactifications of $\mathcal{M}(M)$.

In this regard, we shall discuss below several conditions based on Gromov's celebrated Precompactness Theorem. In addition, in this setting also powerful methods and theorems from the theory of convergence and collapsing of Riemannian manifolds can be employed.

We will then examine in more detail related results and questions in the realm of positive sectional or Ricci curvature, especially under pinching conditions. Here, we especially focus on ones that appear in or arise from the work of X. Rong and his collaborators.

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2 A metric approach to moduli spaces of Riemannian metrics

To approach the investigation of isometry classes of metrics and their moduli spaces' potential compactifications via Gromov-Hausdorff distance, let us first discuss some curvature and other geometric conditions that will ensure precompactness with respect to those, so that we will always be able to work with and compare to limits[2]. As already mentioned, for simplicity we will only consider closed manifolds, but notice again that everything described in this present context will allow for obvious pointed and/or equivariant generalisations.

For a closed smooth manifold M let $\mathcal{ISO}(M, C, L)$ denote the set of isometry classes of Riemannian metrics g on M which satisfy for a given set of geometric constraints C and a given nonnegative real number L an inequality of the form $C(g) \leq L$. Here we want our conditions C to be chosen in such a way that

1. $\mathcal{ISO}(M, C, L)$ is precompact with respect to the Gromov-Hausdorff distance, and such that

2. for every Riemannian metric g on M there exists some real number L with $g \in \mathcal{ISO}(M, C, L)$.

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[1] Using pointed and/or equivariant Gromov-Hausdorff convergence, the same reasoning does, of course, also apply to non-compact and/or equivariant settings.

[2] The roots of what follows actually date back a long time ago and emerged from inspiring discussions with Anton Petrunin. Spasibo, Tosha!

As specific examples for such constraint conditions C we mention[3]:

$max\{1/vol(g), diam(g), \|sec(g)\|\}$	(Cheeger (C) conditions)
$max\{diam(g), -Ric(g)\}$	(Gromov (G) conditions)
$max\{diam(g), \|sec(g)\|\}$	(Cheeger-Fukaya-Gromov (CFG) conditions)
$max\{1/vol(g), diam(g), -sec(g)\}$	(Grove-Petersen (GP) conditions)
$max\{1/vol(g), diam(g), -Ric(g)\}$	(Cheeger-Colding (CC) conditions)
$max\{diam(g), \|Ric(g)\|\}$	(Anderson (A) conditions)

Remark 1 Notice that Gromov, Cheeger-Fukaya-Gromov, and Anderson conditions do allow for collapsing to take place, whereas the other three conditions do not.

Let us now take for fixed $L > 0$ the Gromov-Hausdorff closure $\overline{\mathcal{ISO}}(M, C, L)$ of $\mathcal{ISO}(M, C, L)$, and define $\overline{\mathcal{ISO}}(M, C)$ as the union of all $\overline{\mathcal{ISO}}(M, C, L)$ over all $L > 0$. This way we obtain an object defined by the conditions C that does not depend on any special values of L, but which has the property that every metric space contained inside it can be approximated in a uniformly controlled way by a sequence of Riemannian metrics g_k on M which satisfy $C(g_k) \leq L$ for some L and all k.

Definition 1 We call two closed smooth maniifolds M and M' with respect to a set of conditions C with the properties (1) and (2) from above C *neighbours* if the intersection $\overline{\mathcal{ISO}}(M, C) \cap \overline{\mathcal{ISO}}(M', C)$ is non-empty, and a metric space $X \in \overline{\mathcal{ISO}}(M, C) \cap \overline{\mathcal{ISO}}(M', C)$ will be called a C *gate space* of M and M'.

For example,

- Manifolds which are Cheeger (C) neighbours are always diffeomorphic because of Gromov's convergence theorem for Cheeger bounds. Thus non-diffeomorphic manifolds do not allow for C gate spaces. Notice, however, that $\overline{\mathcal{ISO}}(M, C)$ in general also contains metrics of less regularity than the smooth ones one started with. Indeed, examples show that in general the limit metrics that occur can only be expected to be of class $C^{1,\alpha}$ with $\alpha < 1$, but not of class $C^{1,1}$.
- Any two almost flat manifolds are Cheeger-Fukaya-Gromov (CFG) neighbours, and the one point space is a corresponding CFG gate space for all of them. On the other hand, by Gromov's celebrated almost flat manifold theorem, the one point space can never show up as a CFG gate space for spheres of dimension greater than or equal to two. Moreover, any flat torus T^k is a CFG gate space for flat tori T^n of higher dimensions, and the former do also constitute a complete list of the gate spaces of the latter.

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[3] The proper names associated to the conditions here are meant to refer to the Riemannian geometers who initiated and/or obtained foundational results in their respective studies.

- On the other hand, any two almost non-negatively curved manifolds are Grove-Petersen (GP) neighbours, and the point space is also a GP gate space for all them.
- Every Aloff-Wallach space $M = M_{pq}$ (see below) can be collapsed with bounded curvature to the six-dimensional flag manifold $X = SU(3)/T^2$. Thus any two such M are CFG neighbours, and $SU(3)/T^2$ is a CFG gate space for all of them.
- Let us consider, instead, now the class of closed two-connected manifolds M of dimension at most $n \in \mathbb{N}$. Then, the π_2 finiteness theorems of Fang-Rong and Petrunin-Tuschmann (compare [10, 25]) imply the following: Any given compact metric space X can serve as a CFG gate space only for finitely many M.

Especially in Cheeger-Fukaya-Gromov theory, i.e., collapsing with bounded curvature and diameter, gate spaces are of great importance since they can be viewed as base spaces of (singular) fiber bundles to which the total spaces collapse, and thus greatly simplifiy the classification issues arising in this context. This can, moreover, also be approached equivariantly via the frame bundles, yielding genuine fiber bundles in this way.

The formalism introduced above (re)produces old and new questions, for example:

Question 1 Can two non-diffeomorphic closed n-manifolds be GP neighbours? In particular, can we distinguish non-diffeomorphic homotopy n-spheres in this way?

Remark 2 This question is actually nothing else but the Grove-Wilhelm Smooth Stability Question, compare [14], saying the following: Suppose a sequence of closed n-manifolds (M_k, g_k) with sectional curvature ≥ 1 converges in the Gromov-Hausdorff topology to an n-dimensional compact Alexandrov space A. Does this imply that for all large k_1 and k_2 the manifolds M_{k_1} and M_{k_2} are diffeomorphic?

Remark 3 It was also shown by Grove and Wilhelm in [14] that a positive solution to Question 1 would imply that the famous Grove-Shiohama Diameter Sphere Theorem, see [12], could be improved from homeomorphism to diffeomorphism. For general dimensions, this important question is so far completely open.

Question 2 Can two closed manifolds with non-isomorphic fundamental groups be Cheeger-Colding (CC) neighbours?

Question 3 In what instances can gate spaces be completely classified?

In the next sections, we will ponder some of the above conditions and related questions now in more detail.

3 Results and Questions in Positive and Positively Pinched Curvature

The study of manifolds of positive sectional as well as positive Ricci curvature has always played a prominent and driving role in global Riemannian geometry, but also turned out to be notoriously difficult. It is a rather sobering fact that since today, at least in the setting of sectional curvature, examples of such manifolds have been rather scarce. Indeed, the list of known examples of closed simply connected manifolds admitting a Riemannian metric of positive sectional curvature comprises so far only the following spaces (compare [33] for further details, notation and references):

- the compact rank one symmetric spaces S^n, $\mathbb{CP}^k$, $\mathbb{CH}^l$ and the 16-dimensional Cayley plane $Ca\mathbb{P}^2$;
- Berger's normal homogeneous spaces of the form $B^7 = SO(5)/SO(3)$ and $B^{13} = SU(5)/Sp(2) \cdot S^1$;
- Wallach's homogeneous flag manifolds $W^6 = SU(3)/T^2$, $W^{12} = Sp(3)/Sp(1)^3$, and $W^{24} = F_4/Spin(8)$;
- the homogeneous Aloff-Wallach spaces $M^7_{p,q} = SU(3)/diag(z^p, z^q, \bar{z}^{p+q})$, where $p \geq q \geq 0$ are relatively prime integers and $(p, q) \neq (1, 0)$;
- Eschenburg's 'twisted' flag, the six-dimensional biqotient $E^6 = SU(3)//T^2$;
- Eschenburg's seven-dimensional biquotients $E^7_{kl} = SU(3)//S^1_{k,l}$ for suitable triples of integers $k = (k_1, k_2.k_3)$ and $l = (l_1.l_2, l_3)$;
- Bazakin's thirteen-dimensional biquotients of the form $B^{13}_q = diag(z^{q_1}, \ldots z^{q_6}) \setminus SU(6)/Sp(3)$ where $q = (q_1, \ldots, q_6)$ are six-tuples of integers with $\Sigma\, q_i = 0$.
- a seven-dimensional two-connected manifold P^7_2 with its positively curved metric being invariant under a cohomogeneity one action by $SO(4)$ and being homeomorphic, but not diffeomorphic to the unit tangent bundle of S^4, constructed independently by Dearricott as well as Grove-Verdian-Ziller (compare [9, 13]).

Recall that a smooth manifold M is called (positively) *δ-pinched* if it admits a complete Riemannian metric g such that for some real number $0 < \delta \leq 1$ its sectional curvature satisfies $\delta \leq sec_g \leq 1$. In this case, δ is called the *pinching constant* of the metric g and constitutes one of the most important geometric invariants of (M, g).

Indeed, for strictly quarter-pinched metrics one has from [6] in particular the Brendle-Schoen Differentiable Sphere Theorem, which improves the classical Berger Klingenberg Topological Sphere Theorem from homeomorphism to diffeomorphism:

Theorem 1 *Let (M, g) be a complete, simply connected n-dimensional Riemannian manifold all of whose sectional curvatures are strictly quarter pinched, i.e., satisfy $1/4 < sec_g \leq 1$. Then M is diffeomorphic to S^n with its standard smooth structure.*

Combining this result with Berger's classical rigidity theorem (compare [2, 3]) and the work of Böhm and Wilking on manifolds with positive curvature operator ([5], one also obtains the following rigidity theorem (for other, as well as a smooth stability version for pinching just below one quarter, see also [16] and [23]):

Theorem 2 *A complete, simply connected n-dimensional Riemannian manifold (M, g) all of whose sectional curvatures are weakly quarter pinched, i.e., satisfy $1/4 \leq sec_g \leq 1$, is either diffeomorphic to S^n with its standard smooth structure or isometric to $\mathbb{CP}^{n/2}$, $\mathbb{CH}^{n/4}$, or the 16-dimensional Cayley plane $Ca\mathbb{P}^2$.*

Remark 4 In particular, there are no quarter-pinched exotic spheres, and if a given closed simply connected manifold M admits a positively curved metric g and M is not diffeomorphic to a compact rank one symmetric space, then the pinching constant of g must be strictly less than one quarter.

Remark 5 Of fundamental importance for the proofs of the two theorems just stated above is that in these and related cases there exist uniform lower bounds for the injectivity radius of quarter-pinched metrics.

Indeed, uniform estimates for the injectivity radius of a positively curved Riemannian manifold in terms of its pinching constant have not only proved to be of great importance in the context of sphere and other classification theorems as above, but also for more general diffeomorphism finiteness and convergence theorems, and let us now turn to this topic by mentioning first the so-called Klingenberg-Sakai conjecture (compare [20]):

Conjecture 1 (Klingenberg-Sakai Conjecture) Let M be a closed simply connected manifold and let $0 < \delta \leq 1$. Then there exists $i_0 = i_0(M, \delta) > 0$ such that the injectivity radius i_g of any δ-pinched metric g on M, i.e., any Riemannian metric with sectional curvature $\delta \leq sec_g \leq 1$, is bounded from below by i_0.

Remark 6 In one of his famous problem lists, S.-T. Yau (compare [32]) has posed a somewhat broader question, namely: Does, in the situation above, there exist a positive number i_0 depending only on δ and the *homotopy type* of M, such that the injectivity radius i_g of any δ-pinched metric g on M is bounded from below by i_0?

In this regard, one may also ask the following even more general

Question 4 Let $0 < \delta \leq 1$ and $\mathcal{M}$ be a given class or set of n-dimensional closed simply connected manifolds M. When does there exist $i_0 = i_0(n, \delta) > 0$ such that for every $M \in \mathcal{M}$ the injectivity radius i_g of any δ-pinched metric g on M is bounded from below by i_0?

Remark 7 One may, indeed, also raise this question for the more general case of positive *Ricci* instead of positive sectional curvature pinching, namely, for the less restrictive curvature condition $Ric_g \geq (n-1)\delta > 0$ and $sec_g \leq 1$.

Here are all answers to Question 4 presently known so far:

Theorem 3 *Uniform lower bounds i_0 for the injectivity radius i_g of a Riemannian metric g with positively δ-pinched sectional curvature on an n-dimensional closed simply connected manifold M^n are known to exist in the following principal cases:*

(i) The dimension $n = 2m$ of M^n is even, and $\delta > 0$ is arbitrary. Then $i_0 \geq \pi$. In particular, i_0 does neither depend on M nor the pinching constant δ nor the dimension $n = 2m$ (Klingenberg [17]).

(The case $n = 2$ was solved before by Pogorelov [26].)

(ii) The dimension n of M^n is odd, and $\delta \geq 1/4 - \epsilon$, where $\epsilon \approx 10^{-6}$ (Abresch-Meyer [1]). Again, here i_0 is independent of M and n and greater than or equal to π.

The exact value of ϵ itself is unknown, but the Berger spheres (see Remark 10 below) show that δ must be bigger than $1/9$ for this estimate to hold.

(For odd dimensions n, the case $\delta > 1/4$ was solved before by Klingenberg [18], and the case $\delta \geq 1/4$ was treated independently before by Cheeger-Gromoll [8] and Klingenberg-Sakai [19].)

(iii) The manifold M^n has finite second homotopy group, its dimension n is arbitrary, and $\delta > 0$ is arbitrary. Here, however, i_0 will in general depend on δ and might also depend on the dimension n (Fang-Rong ([10]) and Petrunin-Tuschmann ([25])).

As to Remark 7, generalising both (*iii*) in Theorem 3 to positive Ricci pinching conditions as well as an earlier result of Burago-Toponogov [7] for three-manifolds, we have

Theorem 4 *For every natural number $n \geq 2$ and every $\delta > 0$ there exists a positive constant $i_0(n, \delta > 0$ such that for all closed simply connected n-dimensional manifolds M^n with finite second homotopy group the following is true: If g is any Riemannian metric on M satisfying $Ric_g \geq (n-1)\delta > 0$ and $sec_g \leq 1$, then its injectivity radius i_g is bounded from below by i_0 (Petrunin-Tuschmann ([25])).*

Remark 8 The existence of uniformly positively sectional curvature δ-pinched collapsing sequences among the Aloff-Wallach spaces - which all have second homotopy groups of infinite order - shows that for small positive $\delta < 1/37$ (compare [27]), there is no chance for Question 4 to hold in general, Therefore, one has to impose extra conditions like the vanishing of the second Betti number on the topology of the manifolds M in question. However, if one fixes the homotopy type, then Yau's question, as well as the original Klingenberg-Sakai Conjecture, are so far still completely open.

Remark 9 On the other hand, consider now the more general conditions of *positive Ricci pinching* $sec_g \leq 1$ and $Rig_g \geq (n-1)\delta > 0$ instead of positive sectional curvature pinching. Then, for the existence of uniform positive lower bounds on the injectivity radius it is actually necessary that the second homotopy groups of the manifolds be finite, even if one fixes their topological type. In fact, in [25] it is shown that there is a sequence of metrics $(g_k)_{k\in\mathbb{N}}$ on $S^2 \times S^3$ which satisfy the bounds $sec_{g_k} \leq 1$ and $Ric_{g_k} \geq 4\delta > 0$, but for which the spaces $(S^2 \times S^3, g_k)$ collapse to $S^2 \times S^2$ as $k \to \infty$.

However, there is still one additional way, namely, a Gromov-Hausdorff theoretical approach, to make further progress on the original Klingenberg-Sakai Conjecture. It was initiated in the joint work of Petrunin and Rong and the present author (compare [24]) and paved the way for the injectivity radius estimates obtained in [10] and [25]. To explain it, notice first that in terms of Gromov-Hausdorff convergence, the original conjecture can be reformulated in the following way:

Conjecture 2 (Gromov-Hausdorff Version of the Klingenberg-Sakai Conjecture) Suppose that a closed simply connected manifold M admits a sequence of Riemannian metrics $(g_k)_{k\in\mathbb{N}}$ with sectional curvature $\lambda \leq sec_{g_k} \leq \Lambda$ such that as

$k \to \infty$ the sequence of metric spaces (M, g_k) Gromov-Hausdorff converges to a compact metric space X of lower dimension. Then $\lambda \leq 0$, i.e., these metrics cannot be uniformly positively pinched).

Let us now make the following definition:

Definition 2 A sequence of metric spaces M_k is called *stable* if there is a topological space M and a sequence of metrics d_k on M such that (M, d_k) is isometric to M_k and the metrics d_k converge as functions on $M \times M$ to a continuous pseudometric.

A first main result from [24] then reads as follows:

Theorem 5 *(Stable Collapse Theorem) Suppose that a closed manifold M admits a stable sequence of Riemannian metrics $(g_k)_{k\in\mathbb{N}}$ with sectional curvatures $\lambda \leq sec_{g_k} \leq \Lambda$, such that as $k \to \infty$ the metric spaces (M, g_k) Gromov-Hausdorff converge to a compact metric space X of lower dimension. Then $\lambda \leq 0$ (that is, these metrics cannot be uniformly positively pinched).*

Equivalently stated, this result says:

Theorem 6 *(Bounded Version of the Klingenberg-Sakai Conjecture) Let M be a closed manifold and d_0 a metric on M, and let $0 < \delta \leq 1$. Then there exists $i_0 = i_0(M, d_0, \delta) > 0$ such that the following holds: The injectivity radius i_g of any δ-pinched d_0-bounded metric g on M, i. e., any Riemannian metric g with sectional curvature $\delta \leq sec_g \leq 1$ and $dist_g(x, y) \leq d_0(x, y)$, is bounded from below by i_0.*

Remark 10 Recall that the Berger spheres constitute an example of a collapse of S^{2m+1} to $\mathbb{CP}^m$ by a continuous one parameter family of Riemannian metrics with positive sectional curvature $0 < sec \leq 1$. The next main result from [24] shows in particular that under the assumption of *positive pinching*, $0 < \delta \leq sec_g \leq 1$, such phenomena cannot occur. Indeed, we also have the following *continuous version of the Klingenberg-Sakai conjecture*:

Theorem 7 *(Continuous Collapse Theorem) Suppose that a closed manifold M admits a continuous one parameter family $(g_t)_{0<t\leq 1}$ of Riemannian metrics with sectional curvature $\lambda \leq sec_{g_t} \leq \Lambda$, such that as $t \to 0$ the family of metric spaces (M, g_t) Gromov-Hausdorff converges to a compact metric space X of lower dimension. Then $\lambda \leq 0$ (that is, these metrics cannot be uniformly positively pinched).*

Remark 11 By results from [25], the Stable Collapse Theorem, the Bounded Version of the Klingenberg-Sakai Conjecture Theorem, and the Continuous Collapse Theorem all continue to hold when the uniform positive pinching of sectional curvature $0 < \delta \leq sec_g \leq 1$ is replaced by *positive Ricci pinching conditions $sec_g \leq 1$ and $Ric_g \geq (n-1)\delta > 0$.*

Notice that these results played a crucial role in the proof of the finiteness theorems and injectivity radius estimates for two-connected manifolds from in [10] and [25].

Question 5 Which kind of other conditions on a positively sectional or Ricci δ-pinched M will guarantee that the Stable or Continuous Collapse Theorem will hold?

Let us put these results into perspective with the preceding section of this paper. Notice that all individual conditions given in Theorems 3 and 4 prevent collapsing in the case of Cheeger-Fukaya-Gromov (CFG) conditions when those are specialised to positive sectional or positive Ricci pinching. In particular, in this case every Gromov-Hausdorff limit of a sequence of δ-pinched metrics on a given closed simply connected manifold will be a manifold of the same dimension with a metric of just slightly less regularity.

Hence, what can be said when for positively δ-pinched metrics on a given manifold collapse phenomena do occur?

Remark 12 By [24], the Continuous Collapse Theorem implies in particular also the following: For $0 < \delta \leq 1$, let $\mathcal{M}^\delta(M)$ be the space of δ-pinched metrics on a given closed manifold M. Suppose that the Klingenberg-Sakai conjecture is false. That is, there exists an m-dimensional manifold M which, for some $\delta > 0$, admits a collapsing sequence of δ-pinched metrics $g_n \in \mathcal{M}^\delta(M)$. We may then assume that the sequence of metric spaces (M, g_n) Gromov-Hausdorff converges to a compact metric space X of lower dimension. Then (compare [24]) there exists an $\epsilon = \epsilon(M, \delta, X) > 0$, such that the intersection of $\mathcal{M}^\delta(M)$ with the Gromov-Hausdorff ϵ-neighborhood of X has an infinite number of connected components. Moreover, for each of these components, its infimum distance to X is positive.

Remark 13 In this regard, let us also note that in [22] Kreck and Stolz gave, using Aloff-Wallach spaces, the first examples of closed simply connected manifolds with positive sectional curvature for which the moduli space of positive sectional curvature metrics is disconnected, and that Krannich, Kupers and Randal-Williams ([21]) showed that the space of Riemannian metrics of positive sectional curvature on a closed manifold can have nontrivial higher rational homotopy groups.

What else can be said about collapsed limits of positively curved or positively pinched manifolds?

In [11], Fukaya made the following seemingly natural guess:

Conjecture 3 Let X be a Gromov-Hausdorff limit of a sequence of uniformly positively pinched n-dimensional simply connected Riemannian manifolds. Then $dim(X) \geq n-1$, i.e., every such limit is of codimension at most one.

However, in [25] this conjecture was refuted by the following counterexamples using positively curved biquotients:

Example 1 There exist sequences $(M^7_k)_{k\in\mathbb{N}}$ of uniformly positively pinched Eschenburg spaces which collapse to a 4-dimensional Alexandrov space X of the form $T^2 \setminus SU(3)/T^2$.

Example 2 There exist sequences $(M_k^{13})_{k\in\mathbb{N}}$ of uniformly positively pinched Basaikin spaces which collapse to a 9-dimensional Alexandrov space X of the form $T^5 \setminus U(5)/(Sp(2)\times S^1)$.

X. Rong obtained in [29] then, however, the following dimension estimate, and Example 1 also shows that it is optimal:

Theorem 8 *If X is the Gromov-Hausdorff limit of a sequence of positively δ-pinched closed simply connected Riemannian n-manifolds, then the dimension of X is bounded from below by $(n+1)/2$.*

On the other hand, the following conjecture from [25], related to Fukaya's, is still open:

Conjecture 4 Let X be a Gromov-Hausdorff limit of a sequence of simply connected n-dimensional Riemannian manifolds with sectional curvatures bounded from below by 1. Assume that X is a Riemannian *manifold* of positive dimension. Then the dimension of X is bounded from below by $n-1$.

Let us mention and discuss two more very interesting and intriguing conjectures in the field of positive sectional curvature and pinching. The first is the so-called uniform pinching conjecture attributed to M. Berger, compare [4] as well as [28]:

Conjecture 5 (Uniform Pinching Conjecture) In every dimension $n \geq 2$ there exists $\delta(n) \in (0,1]$ with the following property: If M is a closed n-manifold which admits a metric of positive sectional curvature, then M does also admit a $\delta(n)$-positively pinched metric.

By uniformization and Hamilton's work, this conjecture is true in dimensions two and three, and otherwise wide open.

Inspired by the fact that by the finiteness theorems of Fang-Rong and Petrunin-Tuschmann [10, 25], there exist only finitely many diffeomorphism types of two-connected positively pinched manifolds, if one specifies the dimension and the pinching constant, X. Rong (see [30]) has proposed the following

Conjecture 6 (Positive Curvature Two-Connectedness Finiteness Conjecture) In every dimension $n \geq 2$ there exist only finitely many diffeomorphism types of closed two-connected n-manifolds which admit a Riemannian metric of positive sectional curvature.

Remark 14 Notice that by [10, 25], a positive solution to the Uniform Pinching Conjecture would imply the Positive Curvature Two-Connectedness Finiteness Conjecture.

On the other hand, there might actually be also counterexamples to both. Namely, in dimension seven there is an infinite family of closed two-connected cohomogeneity one manifolds $(P_k)_{k\in\mathbb{N}}f$ of pairwise distinct homotopy type which are indeed very hot candidates for admitting metrics of positive sectional curvature (compare [15]). (Here $P_1 = S^7$ and P_2 is the manifold homeomorphic to the unit tangent bundle of $S4$ on which positive sectional curvature metrics have been constructed by Dearricott and Grove-Verdiani-Ziller (see [9, 13]). Indeed, by the work of Grove, Wilking and Ziller (compare [15]), all P_k could even admit metrics of positive sectional curvature that are invariant under the respective cohomogeneity one actions. This leads to the following question and remark:

Question 6 Does there exists an infinite subsequence $(P_l)_{l\in\mathbb{N}}$ inside the sequence of cohomogeneity one seven-manifolds P_k above for which each member does admit (some, not necessarily invariant) metric of positive sectional curvature?

Remark 15 An affirmative answer would contradict the Positive Curvature Two-Connectedness Finiteness Conjecture. Moreover, by the diffeomorphism finiteness theorems from [10, 25], the respective pinching constants of the P_l must necessarily converge to zero as $l \to \infty$. In particular, this would also refute the Uniform Pinching Conjecture.

In order to find new manifolds of positive sectional curvature and approach Berger's as well as Rong's conjecture, much further work has to be done. It is, for example, in general dimensions still totally unclear if a nonnegative sectional curvature metric on a given closed simply connected manifold could be deformed into a positively curved one. But understanding the geometry of the P_k spaces more closely seems to be one of the most promising and feasible ways to achieve this goal. Much further study should therefore be devoted to them.

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