Moduli Spaces of Nonnegatively Curved Metrics on Closed Manifolds and RCD spaces

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Abstract We survey and present recent (and partially slightly improved) results and open questions about the global topological properties of moduli spaces of nonnegatively curved metrics on closed manifolds and RCD spaces as well as of moduli spaces of Ricci flat metrics on $K3$ surfaces and their relatives.

1 Introduction

Consider a smooth manifold with a Riemannian metric satisfying some sort of geometric constraint like, for example, positive scalar curvature, non-negative Ricci or negative sectional curvature, being Einstein, Kähler, Sasakian, etc. A natural question to ponder is then what the space of all such metrics does look like. Moreover, one can also study this question for the corresponding moduli spaces of metrics, i.e., quotients of the former by the diffeomorphism group of the manifold, acting by pulling back metrics. These spaces are customarily equipped with the topology of smooth convergence on compact subsets and the corresponding quotient topology, respectively, and their topological properties then provide the right means to measure 'how many' different metrics, and geometries, respectively, the given manifold actually does exhibit.

The history of the subject as a whole indeed goes back more than one hundred years, and since H. Weyl's early result on the connectedness of the space of positive Gaussian curvature metrics on S^2 ([20]) and the foundings of Teichmüller, infinite-dimensional manifold and Lie group theory, uniformization and geometrization, the study of spaces of metrics and their corresponding moduli has been a topic of interest for differential geometers, global and geometric analysts and topologists alike.

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Especially in the last decade there has been intensive activity and substantial further progress on these topics, and, moreover, in a similar vein, and with appropriate modifications, one has also set out to investigate analogous questions for more general metric spaces than Riemannian manifolds.

The present article is not intended to cover this vast subject as a whole. Instead, it merely focuses on recent developments around nonnegatively curved metrics on closed Riemannian manifolds and RCD spaces, as well as of moduli spaces of Ricci flat metrics on $K3$ surfaces, and its remaining parts are structured as follows: Section 2 discusses several general results about topological properties of moduli spaces of closed Riemannian metrics with nonnegative sectional or nonnegative Ricci curvature. Section 3 is devoted to what is known by D. Degen for Ricci flat metrics on $K3$ surfaces, and in section 4, we discuss first results in the realm of RCD spaces obtained by A. Mondino and D. Navarro.

For further general background on the field as a whole, as well as results on spaces and moduli spaces of metrics with nonnegative sectional curvature on non-compact manifolds, or metrics with positive scalar, Ricci or sectional curvature, or metrics with negative sectional curvature, we refer to the monograph [19]. For more details on the proofs of the results presented in the sequel, as well as related matters, we ask the interested reader to consult the original literature cited here.

2 Moduli Spaces of Nonnegative Sectional and Ricci Curvature Metrics

2.1 Disconnectedness Results

Based on early seminal work of Kreck-Stolz coming up with an index invariant for distinguishing path components of moduli spaces of positive scalar curvature metrics ([13]) using the famous Atiyah-Patodi-Singer Index Theorem, a first general disconnectedness result for moduli spaces of nonnegative sectional curvature metrics on manifolds in an infinite range of dimensions was obtained by Dessai, Klaus and the author of this article in 2018 in [7] as follows:

Theorem 1 *In each dimension $4k+3$, $k \geq 1$, there exist infinite sequences of closed smooth simply connected manifolds of pairwise distinct homotopy type for which the moduli space of Riemannian metrics with nonnegative sectional curvature has infinitely many path components.*

Remark 1 The theorem holds true verbatim when 'nonnegative sectional curvature' is replaced by 'positive Ricci curvature', and it complements here also an earlier result of D. Wraith ([22]) who showed that for any homotopy sphere Σ^{4k+3}, $k \geq 1$, bounding a parallelizable manifold, the moduli space of metrics with positive Ricci curvature has an infinite number of path components.

Remark 2 The manifolds figuring in the theorem are total spaces of principal circle bundles over $\mathbb{C}P^{2n} \times \mathbb{C}P^1$ and can, inspired by a simple observation from [11], also be viewed as quotients of the product of round spheres $S^{4n+1} \times S^3$ by free isometric circle actions. Equipping them with the corresponding submersion metrics, one shows that in any fixed dimension $4n + 3$ there exist among these Riemannian manifolds infinite sequences with the following properties: these manifolds belong to a fixed diffeomorphism type and their metrics have (simultaneously) nonnegative sectional and positive Ricci curvature, but pairwise distinct so-called s-invariants (compare [13]), which tell apart the components of the moduli spaces. Since there are infinite sequences of manifolds of pairwise distinct homotopy types with these properties in each given relevant dimension, this proves the theorem.

So, what about other dimensions?

First of all, now using relative eta-invariants in conjunction with the Atiyah-Patodi-Singer Index Theorem, in [6] Dessai and González-Álvaro showed that the moduli space of metrics of nonnegative sectional curvature on each of the – up to orientation-preserving diffeomorphism – four homotopy $\mathbb{RP}^5$ has infinitely many path components. Moreover, in [5] Dessai recently obtained, via a closely related approach, the following general result:

Theorem 2 *In every dimension $4k + 1$, $k \geq 2$, there are infinite sequences of closed manifolds with pairwise nonisomorphic integral cohomology for which the moduli space of metrics of nonnegative sectional curvature has infinitely many path components.*

Remark 3 As in Theorem 1, this result also holds true when 'nonnegative sectional curvature' is replaced by 'positive Ricci curvature'. However, in order to ensure that the invariants used can detect different path components, it is now crucial that the manifolds in question all do have a *non-trivial* (though finite) fundamental group.

Remark 4 In Dessai's theorem, the manifolds admit the structure of total spaces of iterated fibre bundles over the base $\mathbb{CP}^1$ and fibres $\mathbb{CP}^{2k-1}$ and the circle S^1, but to obtain metrics of nonnegative sectional (and positive Ricci curvature) on them, they are simultaneously also exhibited as quotients of products of standard spheres by an isometric and free action of the group $S^1 \times \mathbb{Z}_2$.

For other related results, see [5] and the further references given there.

It remains an open question whether these results also hold true in the dimensions not covered so far, and the first basic problem to be solved to answer it consists in constructing new invariants to distinguish path components in the moduli space of nonnegatively curved metrics here. Also, will it be worthwhile to look for this behalf instead on metrics of nonnegative Ricci curvature?

On the other hand, there is also the following result of Wiemeler and the present author from [18] (compare there Theorem 1.3 as well as Remark 1.9):

Theorem 3 *Let M be a simply connected closed smooth manifold which admits a Riemannian metric with nonnegative Ricci curvature, and let T^k denote a k-dimensional standard smooth torus, where $k \in \mathbb{N}$.*

Then the moduli space of Riemannian metrics with nonnegative Ricci curvature on $M \times T^k$ has at least as many path components as the moduli space of metrics with nonnegative Ricci curvature on M.

Moreover, this statement carries over from the torus T^k to all other closed flat manifolds, and holds as well when nonnegative Ricci is replaced by nonnegative sectional curvature.

Taking products of the examples given by Theorem 1 with closed flat manifolds, we thus obtain:

Corollary 1 *In every dimension $n \geq 7$ there exist infinite sequences of closed smooth manifolds of pairwise distinct homotopy type for which the moduli space of Riemannian metrics with nonnegative sectional curvature has infinitely many path components.*

Outside the dimensions $4k + 3$, $k \geq 1$ given by Theorem 1 and $4k + 1$, $k \geq 2$ given by Theorem 2, the manifolds in Corollary 1 are thus bound to have infinite fundamental group. Thus, can the corollary be here yet strengthened to manifolds with finite or trivial fundamental group?

2.2 Higher Homotopy Groups

Opposed to the "π_0" case of path components, indeed very little is known so far about higher homotopy (or (co)homology) groups of moduli spaces of Riemannian metrics with nonnegative sectional or nonnegative Ricci curvature. Here we will treat the general case, and what is, moreover, known for Ricci flat metrics on $K3$ manifolds will be discussed in the next section. Now, to the best of the author's knowledge, in this context the following theorem from [18] is so far the only result about the homotopy (and cohomology) groups of the moduli spaces of Ricci nonnegative metrics on closed manifolds:

Theorem 4 *Let M be a simply connected closed smooth manifold which admits a metric with nonnegative Ricci curvature and let T be a torus of dimension $k \geq 4$, $k \neq 8, 9, 10$. Then the moduli space $\mathcal{M}_{\mathrm{Ric}\geq 0}(M \times T)$ of nonnegatively Ricci curved metrics on $M \times T$ has non-trivial higher rational cohomology groups and non-trivial higher rational homotopy groups.*

Remark 5 When the torus T figuring there is replaced by any other closed flat manifold F, the theorem and its conclusions will carry over as well, provided that the moduli space of flat metrics on F does exhibit corresponding non-trivial topological properties as the moduli space of flat metrics on T (compare Remark 1.9 in [18]).

Remark 6 The conclusion of the theorem carries over verbatim to moduli spaces of nonnegative sectional curvature if M admits a Riemannian metric of this kind.

Remark 7 For $n = 4$ it is the third rational cohomology group (rational homotopy group, respectively) of the moduli space which is always non-trivial. In the case where $n > 4$ it is the fifth rational cohomology group (rational homotopy group, respectively) which is non-trivial. For further details, we refer to [18].

Remark 8 The proof of Theorem 4 hinges on two essential ingredients. The first consists in using the Cheeger-Gromolll Splitting Theorem to construct retraction mappings from moduli spaces of nonnegatively Ricci curved manifolds, which are diffeomorphic to certain products, to their corresponding factors, and the second in gaining detailed knowledge about the topology of the moduli space of flat metrics on n-dimensional tori. Indeed, by ([18], Proposition 5.5), as a result of independent interest, in the latter regard we have:

Theorem 5 *If T^n is an n-dimensional standard torus, then the following holds:*

1. *The moduli space $\mathcal{M}_{\mathrm{sec}=0}(T^n)$ of flat metrics on T^n is simply connected.*
2. *If $n = 1, 2, 3$, then $\mathcal{M}_{\mathrm{sec}=0}(T^n)$ is contractible.*
3. *If $n = 4$, then $\pi_3(\mathcal{M}_{\mathrm{sec}=0}(T^n)) \otimes \mathbb{Q} \cong \mathbb{Q}$.*
4. *If $n > 4$, and $n \neq 8, 9, 10$, then $\pi_5(\mathcal{M}_{\mathrm{sec}=0}(T^n)) \otimes \mathbb{Q} \cong \mathbb{Q}$.*

Remark 9 The proof makes crucial use of the fact that by results of Wolf, compare [21], the space $\mathcal{M} = \mathcal{M}_{\mathrm{sec}=0}(T^n)$ can be identified with the biquotient space

$$O(n)\backslash\, GL(n, \mathbb{R})\, / GL(n, \mathbb{Z}).$$

Considering manifolds which are products of simply connected round spheres with a flat four-torus, Theorem 4 and Theorem 5 in particular yield:

Corollary 2 *In every dimension $n \geq 4, n \neq 5$ there exist closed smooth manifolds M^n for which the third rational cohomology group and the third rational homotopy group of the moduli space of metrics with nonnegative sectional curvature on M^n are non-trivial.*

Remark 10 Independently of the dimension n, all manifolds in the corollary can thus be chosen to satisfy that their fundamental group is isomorphic to $\mathbb{Z}^4$, but it is an interesting open question if there are even simply connected closed manifolds for which the conclusion of the corollary holds. Compare in this regard, in particular, also Corollary 5 below.

Remark 11 On the other hand, in [8, 9] (see also the next section of this paper), it was in particular shown that the moduli space of flat metrics on the product manifold of a circle S^1 and a Klein bottle retracts onto the circle and that there is a closed four-manifold whose moduli space of flat metrics retracts onto a three-punctured sphere. With this information at hand, and in conjunction with the retraction arguments used in the proof of Theorem 4, and Remark 5, one does obtain in a similar way as in Corollary 2 also the following statement:

Corollary 3 *In every dimension $n \geq 3$ there exist closed smooth manifolds M^n for which the fundamental group of the moduli space of metrics with nonnegative sectional curvature on M^n has infinite order.*

3 Moduli Spaces of Ricci Flat Metrics on $K3$ Surfaces

It is well known that very closed Ricci flat manifold of dimension at most three is finitely covered by a flat torus, but in dimension four, all known examples are finitely covered either by the flat torus T^4 or, in the case the metric is Ricci flat but not flat, by the $K3$ manifold. What is this exactly? Traditionally, algebraic or complex geometers define a $K3$ *surface* as a simply connected compact complex analytic surface with trivial first Chern class, and there are a multitude of examples of these, comprising Kummer and some elliptic surfaces, and, of course, the classical Fermat quartic given in homogeneous coordinates as $\{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset \mathbb{CP}^3$.

However, up to diffeomorphism, there is only one smooth manifold underlying all $K3$ surfaces which henceforth will be called the $K3$ *manifold* K.

Notice that $K3$ surfaces enjoy remarkable differential geometric properties. Indeed, if K is the $K3$ manifold and g a Riemannian metric on K, then g is Ricci flat iff g is Einstein iff g has vanishing scalar curvature iff g has nonnegative scalar curvature iff g is hyperkähler.

The starting point for investigating the topological structure of the corresponding (all-isomorphic) moduli spaces is then the following classical Torelli type theorem, compare [3], [14], [17], which in the present context may be stated as follows:

Theorem 6 *The moduli space of unit volume Einstein metrics on the $K3$ manifold K is homeomorphic to an open and dense subspace of the biquotient space*

$$\Gamma \backslash O(3, 19) \, / (O(3) \times O(19)),$$

where Γ is a discrete subgroup of $O(3, 19)$.

Remark 12 By work of Kobayashi-Todorov ([12]), points in the above biquotient which are not associated to a smooth Einstein metric, are known to naturally correspond to certain Ricci flat orbifold metrics. Moreover, Anderson subsequently proved that this space is also isomorphic to the L^2 completion of the subspace of smooth Einstein metrics on K, compare [1].

In his Ph.D. thesis [4], the present author's former student David Degen used these facts as a starting point to prove, among others, the following results:

Theorem 7 *The moduli space of Ricci flat metrics on the $K3$ manifold K is simply connected and its second rational homotopy group is isomorphic to $H^2(\Gamma, \mathbb{Q}) \oplus \mathbb{Q}$,* *where Γ is an arithmetic subgroup of $O(3,19)$ given by the automorphism group of the lattice $H^2(K,\mathbb{Z})$ with its cup-pairing. In particular, its second Betti number is always positive.*

Theorem 8 *The moduli space of Ricci flat metrics, including orbifold metrics, on the $K3$ manifold K is simply connected and its fourth Betti number is at least* 1.

Crossing the $K3$ manifold with flat tori and simply connected round spheres, respectively, and as in section 2 again using the general retraction arguments established in the proof of Theorem 4 (compare Remark 8), Theorem 7 therefore yields:

Corollary 4 *In every dimension $n \geq 4$ there exist closed smooth manifolds M^n for which the second rational homotopy group of the moduli space of Ricci flat metrics on M^n is non-trivial.*

Corollary 5 *In every dimension $n \geq 4, n \neq 5$ there exist simply connected closed smooth manifolds M^n for which the second rational homotopy group of the moduli space of nonnegatively Ricci curved metrics on M^n is non-trivial.*

Remark 13 In view of these results, a natural question is then in how far they also possess analogues for other manifolds admitting Ricci flat metrics. In particular, the methods developed by Degen should extend to a corresponding analysis of the moduli spaces of Ricci flat metrics on Enriques surfaces. The latter can be defined as compact complex surfaces with fundamental group isomorphic to $\mathbb{Z}_2$ and trivial real first Chern class. They all carry Ricci flat Riemannian metrics and have a $K3$ surface as universal cover, and, moreover, viewed as smooth manifolds any two Enriques surfaces are diffeomorphic.

Remark 14 The moduli spaces of sectional curvature flat metrics on closed manifolds are, except from the case of tori, also still far from being understood in full generality. However, in dimension three and four one has complete information thanks to the work of Karla García, another former student of mine, as well as of her and Oscar Palmas, see [8], [9], and [10].

4 Moduli Spaces of Nonnegatively Curved RCD Spaces

Let us now leave the realm of smooth metrics and look instead on ways to study moduli spaces of singular ones with given kinds of curvature constraints. What comes first to mind are Alexandrov[1] as well as RCD space metrics, and it is the latter we would like to discuss in some further detail in this section.

[1] Indeed, using tools from convex surface theory, for the two-sphere, Belegradek has given a detailed study of the moduli space of metrics with nonnegative curvature in the Alexandrov sense in [2].

Recall that RCD(K,N) spaces are a special class of metric measure spaces (for both notions, see, e.g., [15]) which may be conceived of as potentially non-smooth spaces with dimension at most $N \in \mathbb{N}$ and Ricci curvature bounded from below by $K \in \mathbb{R}$ in a synthetic sense. Their importance in Riemannian and metric geometry stems in particular from the fact that they arise as Ricci limit spaces, i.e., (measured) Gromov-Hausdorff limits of Riemannian manifolds whose Ricci curvature is uniformly bounded from below.

The study of moduli spaces of RCD(0,N) structures (see directly below) has only recently been set up in a foundational paper by Mondino and Navarro, compare [15]. Here, the authors start out by defining that a topological space X is said to admit an *RCD(K,N) structure* if there exists a metric distance function d and a measure μ on X such that the topology induced by d coincides with the topology of X, the measure μ has full support on X, and the couple (d, μ) turns the metric measure space (X, d, μ) into an RCD(K,N) space.

Now, given a compact topological space which admits an RCD(K,N) structure, the *moduli space of RCD(K,N) structures on* X is defined as the quotient space

$$\mathcal{M}_{\mathrm{K,N)}}(\mathrm{X}) := \{\mathrm{RCD(K,N)\ structures\ on\ X}\}/\!\sim$$

of the set of all RCD(K,N) structures on X by the equivalence relation "$\sim$" of isomorphism of metric measure spaces.

Then, in part along the lines and main constructions of [18], for a given RCD(0,N) structure (d, μ) on a compact topological space X, the authors show how to associate notions of lifts (to universal covers), souls and a compact flat orbifold, the Albanese variety, to it, and prove that these depend continuously on the pair (d, μ).

This implies, in particular, that if M is a k-dimensional closed flat manifold, then the moduli space $\mathcal{M}_{(0,\mathrm{N+k})}(X \times M)$ retracts onto the moduli space of flat metrics on M, and by the results from [8] and [18] the authors of [15] obtain the following RCD(0,N) analogues to Theorem 4 and Corollaries 2 and 3 above:

Theorem 9 *Let $N \in \mathbb{N}$ and X be a compact topological space which admits an RCD(0,N) structure and whose revised fundamental group $\bar{\pi}_1(X)$ vanishes (for the latter, compare Theorem 1.2 of [15]). Moreover, let Y be either the product of S^1 and a Klein bottle, or a torus of dimension $k \geq 4$ and $k \neq 8, 9, 10$. Then the moduli space $\mathcal{M}_{(0,N+dim(Y))}(X \times Y)$ has non-trivial higher rational homotopy groups.*

Corollary 6 *There exists for every $N \geq 3$ (resp. $N \geq 4/N \geq 5$) a compact topological space X such that $\mathcal{M}_{(0,N)}(X)$ is not simply connected (resp. has non-trivial third rational homotopy group / non-trivial fifth rational homotopy group).*

Remark 15 In [16], D. Navarro gives a complete list of all compact topological spaces admitting an RCD(0,2) structure and describes for each of them the associated moduli space of RCD(0,2) structures. It turns out that all of them are contractible, which nicely complements as well as contrasts the above corollary.

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