Exercise Sheet No. 4
Advanced Mathematics I

Exercise 16: Let the sequence \((x_n)_n\) be recursively defined by
\[ x_1 = 0, \quad x_{n+1} = \frac{1}{2} (1 - x_n^2), \quad n \in \mathbb{N}. \]

(a) Compute \(x_j\) explicitly for \(j \in \{1, 2, 3, 4\}\) and show by mathematical induction that the following estimation is true for all \(n \in \mathbb{N}\):
\[ 0 \leq x_n \leq \frac{1}{2}. \]

(b) Conclude that the inequality \(|x_{n+1} - x_n| \leq \frac{1}{2} |x_n - x_{n-1}|\) holds for all \(n \in \mathbb{N}, n \geq 2\), and prove by mathematical induction that
\[ |x_{n+1} - x_n| \leq \left( \frac{1}{2} \right)^n, \quad n \in \mathbb{N}. \]

Exercise 17: Let the sequences \((a_n)_n\) and \((b_n)_n\) be defined by:
\[ a_n = n^2 + 1, \quad b_n = \frac{n^3 + n^2 + 3n + 1}{n^4 + n^2 - 3}, \quad n \in \mathbb{N}. \]
Compute the first 6 terms of each sequence \((a_n)_n\) and \((b_n)_n\). Which of the sequences are bounded? Give a formal proof of your answer.

Exercise 18: Compute the limit of each of the following sequences
(a) \(a_n = \frac{n^4 - 2}{n^4 + 4} + \frac{n^3 (3 - n^2)}{n^2 + 1}\)
(b) \(b_n = (1 + (-3)^n) \cdot \left( \frac{10^n}{n!} - \frac{3n^2 + 1}{2n + 1} \right)\)
(c) \(c_n = \sqrt[3]{17} \cdot 2^n \left( \sqrt{n + 1} - \sqrt{n} \right)\).

Exercise 19: Calculate the limits of the complex sequences
(a) \(a_n = 2 + \frac{3}{4i}n + \left( \frac{1}{2} + \frac{1}{3}i \right)^n, \quad n \in \mathbb{N}, \quad (b) \ b_n = \frac{(3i n + 1)(2n + i)}{\sum_{k=1}^{n} ik}, \quad n \in \mathbb{N}.\)

Exercise 20: Consider the sequence \((a_n)_n\) with \(a_n = \frac{n-1}{n+1}, n \in \mathbb{N}\). Find an index \(N\) such that \(|a_n - 1| < \varepsilon\) for every \(n \geq N\), when
(a) \(\varepsilon = \frac{1}{10}\) (b) \(\varepsilon = \frac{1}{1000}\), (c) \(\varepsilon > 0\) is arbitrary.
(d) Does the sequence \((a_n)_n\) converge? If so, what is the limit?

Due date: Your written solutions are due at 14:00 on Tuesday, 20 November, 2018. Please submit them at the beginning of the problem session or in the box in J101 (note the box will be emptied before the problem session).
Website: For detailed information regarding this course visit the following web page:

http://www.math.kit.edu/iag6/edu/am12018w/en