① Permutations and Combinations

- Multinomial Coefficients
- Twelvefold Way
- Cycle Decompositions

② Inclusion-Exclusion-Principle & Möbius Inversion

- PIE
- Möbius Inversion Formula

③ Generating Functions

- Ordinary and Exponential
- Newton's Binomial Theorem
- Reccurence Relations

④ Partitions

(Non-Crossing) Partitions of $[n]$

Ferrer Diagrams

Standard Young Tableaux

⑤ Partially Ordered Sets

(Symmetric) Chain Partitions

Dimension

Posets Between Two Lines

⑥ Designs

Existence/ Non-Existence

Steiner Triple Systems etc.

Latin Squares

$$\binom{n}{r_1,\ldots,r_k} = \frac{n!}{r_1!\cdot\ldots\cdot r_k!} \qquad \text{where } n = r_1 + \ldots + r_k$$

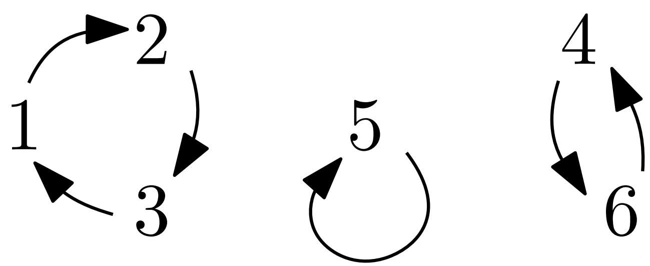
$\#$ n-permutations of multiset $M = \{r_1 \cdot t_1, \ldots, r_k \cdot t_k\}$

Multinomial Theorem:

$$(x_1 + \ldots + x_k)^n = \sum_{r_1 + \ldots + r_k = n} \binom{n}{r_1,\ldots,r_k} x_1^{r_1} \cdots x_k^{r_k}$$

n balls	k boxes	≤ 1 per box	≥ 1 per box	arbitrary
U	**L**	$\binom{k}{n}$	$\binom{n-1}{k-1}$	$\binom{n+k-1}{k-1}$
L	**U**	1	$s_k^{II}(n)$	$\sum_{i=1}^{k} s_i^{II}(n)$
L	**L**	$\binom{k}{n}n!$	$s_k^{II}(n)k!$	k^n
U	**U**	1	$p_k(n)$	$\sum_{i=1}^{k} p_i(n)$

$$\pi = (123)(5)(46) = 231654 =$$



- unique partition into **disjoint** cycles
- exactly k cycles $\longrightarrow$ $s^I_k(n)$ permutations
- no trivial cycles $\longrightarrow$ derangements

② Inclusion-Exclusion-Principle & Möbius Inversion

PIE

Möbius Inversion Formula

- properties $P_1, \ldots, P_m$
- $N(S) = \{x \mid x \text{ has } P_i \text{ for all } i \in S\}$ $\qquad S \subseteq [m]$

Then $\sum_{S \subseteq [m]} (-1)^{|S|} |N(S)|$ elements have none of the properties.

Usage:

Define P_i $\longrightarrow$ Bound $|N(S)|$ $\longrightarrow$ Apply PIE

② Inclusion-Exclusion-Principle & Möbius Inversion

PIE | Möbius Inversion Formula

Stronger PIE

$$g(A) = \sum_{S \subseteq A} f(S) \quad \Rightarrow \quad f(A) = \sum_{S \subseteq A} (-1)^{|A|-|S|} g(S)$$

Möbius Inversion

$$g(n) = \sum_{d|n} f(d) \quad \Rightarrow \quad f(n) = \sum_{d|n} \mu(d) g(\tfrac{n}{d})$$

General Posets

$$g(y) = \sum_{x \leq y} f(x) \quad \Rightarrow \quad f(y) = \sum_{x \leq y} \mu(x,y) g(x)$$

③ Generating Functions

Ordinary and Exponential | Newton's Binomial Theorem | Reccurence Relations

$$F(x) = \sum_{n \geq 0} f_n x^n$$

unlabeled objects

$$A(x) \cdot B(x) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x_n$$

$$G(x) = \sum_{n \geq 0} g_n \frac{x^n}{n!}$$

labeled objects

$$A(x) \cdot B(x) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) \frac{x_n}{n!}$$

Newton's Binomial Theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \qquad n \in \mathbb{R} - \{0\}$$

$$F(x) = 1 + x \cdot F(x)^2 \quad \Rightarrow \quad F(x) = \sum_{n \geq 0} \frac{1}{n+1}\binom{2n}{n} x^n$$

Catalan Numbers: $C_n = \frac{1}{n+1}\binom{2n}{n}$

③ Generating Functions

Ordinary and Exponential | Newton's Binomial Theorem | **Reccurence Relations**

reccurence

$$p(A)f = g$$

initial values

$$f(0) = f_0, \ldots, f(k) = f_k$$

homogeneous $(g = 0)$**:**

$$p(A) = (A - r)^k \;\Rightarrow\; f(n) = c_1 r^n + \cdots + n^{k-1} c_k r^n$$

$$p(A) = p_1(A) \cdot p_2(A) \;\Rightarrow\; f(n) = f_1(n) + f_2(n)$$

non-homogeneous $(g \neq 0)$**:**

$f_0(n)$ gen. sol. of hom. system $\longrightarrow$ $f_1(n)$ particular sol. of non-hom. system $\longrightarrow$ $f = f_0 + f_1$

④ **Partitions**		
(Non-Crossing) Partitions of $[n]$	Ferrer Diagrams	Standard Young Tableaux

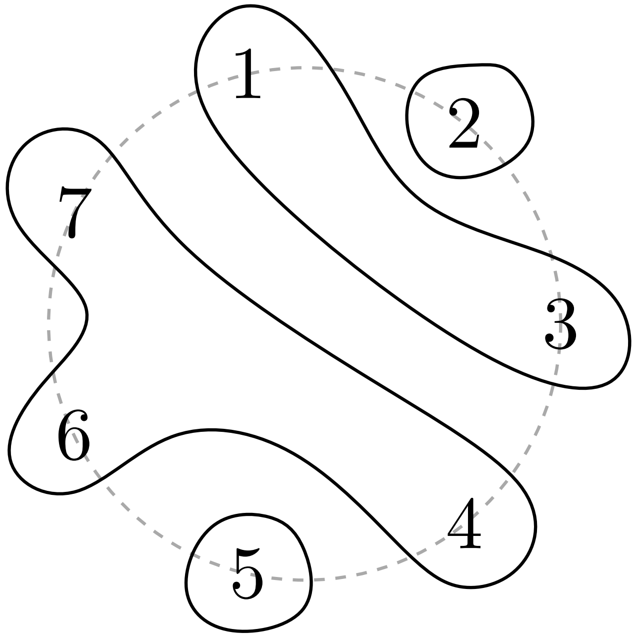
Bell Numbers $$B_n = \sum_{k=0}^{n} s_k^{II}(n)$$

\# ways to split n persons in groups

Non-Crossing Partitions

counted by Catalan numbers

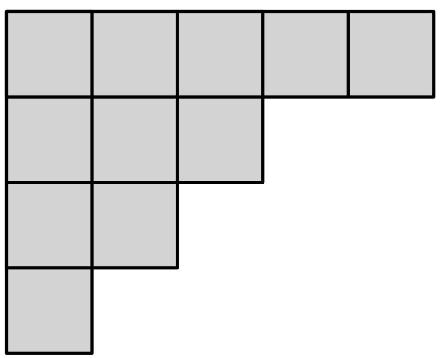
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$



④ Partitions

(Non-Crossing) Partitions of $[n]$	Ferrer Diagrams	Standard Young Tableaux

$$p(n) = \#\text{ ways to write } n \text{ as a sum}$$



$$11 = 5 + 3 + 2 + 1$$

Thm. $p_{odd}(n) = p_{dist}(n)$

Pentagonal Numbers ω_k

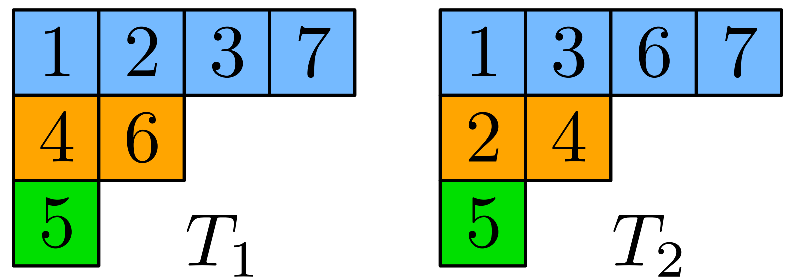
$$p_d^{even}(n) - p_d^{odd}(n)$$

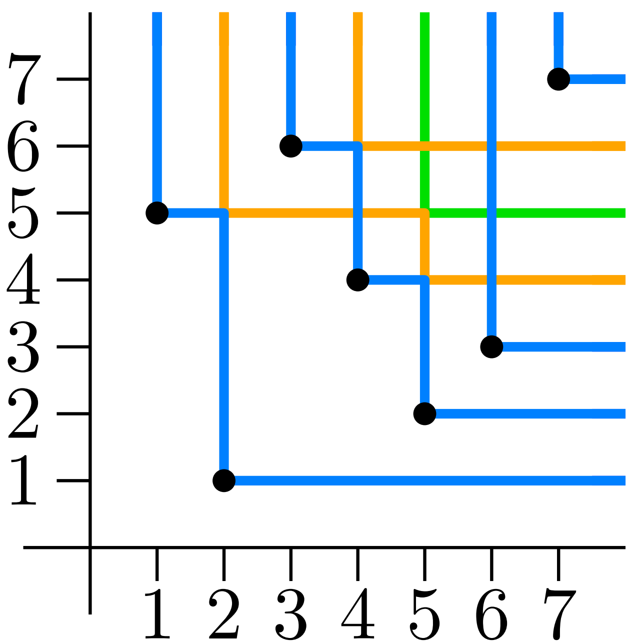
$$= \begin{cases} (-1)^k & \text{if } n = \omega_k \\ 0 & \text{otherwise} \end{cases}$$

④ Partitions

(Non-Crossing) Partitions of $[n]$ | Ferrer Diagrams | Standard Young Tableaux

$\pi = 5264237$ $\overset{\textbf{RSC}}{\longleftrightarrow}$





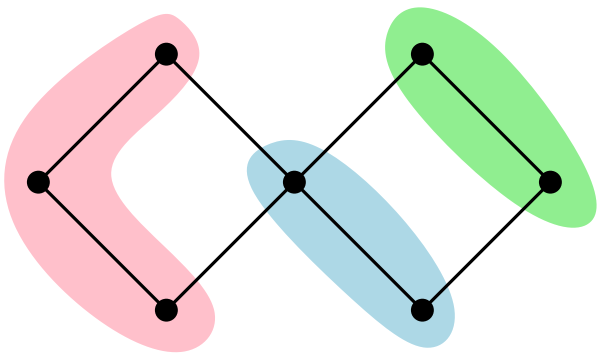
Hook Length Formula

$$t(\lambda) = \frac{n!}{\prod_{(i,\,j)} |h_{i,j}|}$$

⑤ Partially Ordered Sets

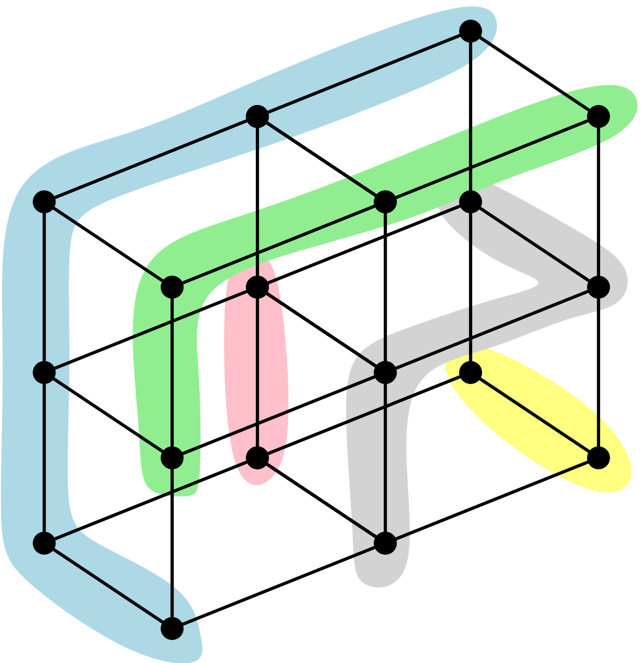
(Symmetric) Chain Partitions | Dimension | Posets Between Two Lines

Dilworth's Theorem



partition into $w(P)$ chains

Multiset Lattices



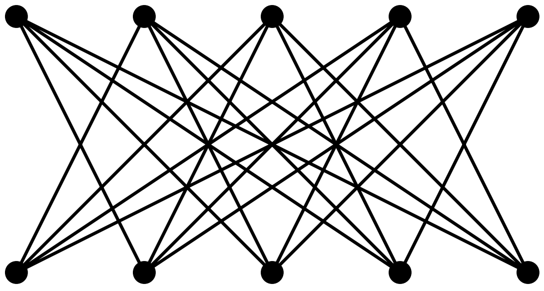
symmetric chain partition

⑤ Partially Ordered Sets

(Symmetric) Chain Partitions | **Dimension** | Posets Between Two Lines

$$\dim(P) = \min\{k \mid P = L_1 \cap \cdots \cap L_k\}$$

Thm: $\dim(P) \leq w(P)$

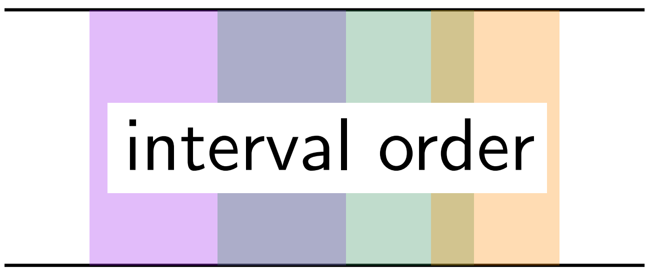


$$\dim(S_n) = w(S_n) = n$$

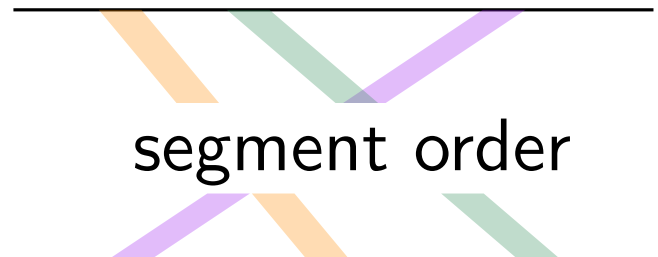
Thm: $\dim(P) \leq 2 \ \Leftrightarrow\ \exists L$ ordering all $1 \oplus 2$

⑤ Partially Ordered Sets

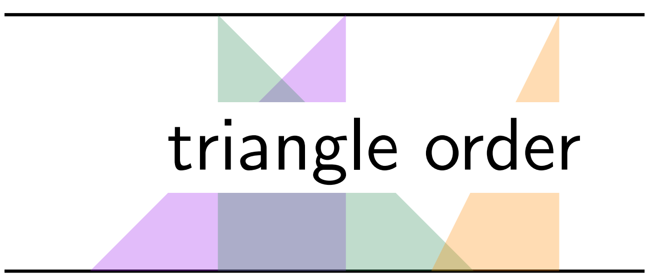
(Symmetric) Chain Partitions | Dimension | **Posets Between Two Lines**



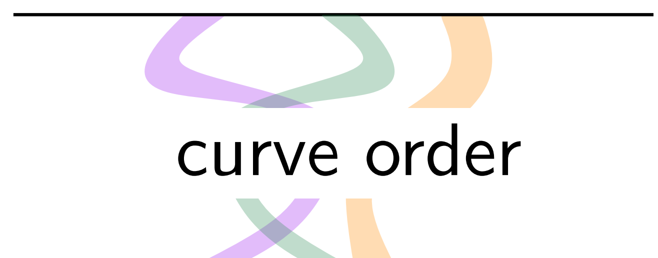
no $2 \oplus 2$



$\dim(P) \leq 2$



$\exists L$ ordering all $2 \oplus 2$



all posets

6 Designs

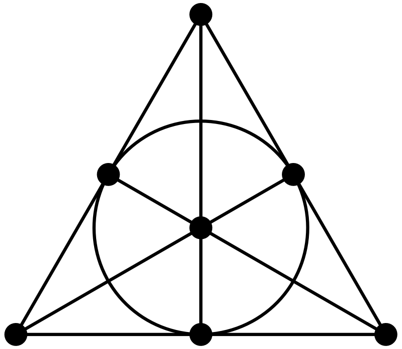
Existence/ Non-Existence	Steiner Triple Systems etc.	Latin Squares

t-(v, k, λ) **Design:** v points, blocks are k-sets, every t-tuple of points in λ blocks

Thm: $\#$ blocks $= \lambda \cdot \binom{v}{t} / \binom{k}{t}$

repetition of i-sets $= \lambda \cdot \binom{v-i}{t-i} / \binom{k-i}{t-i}$

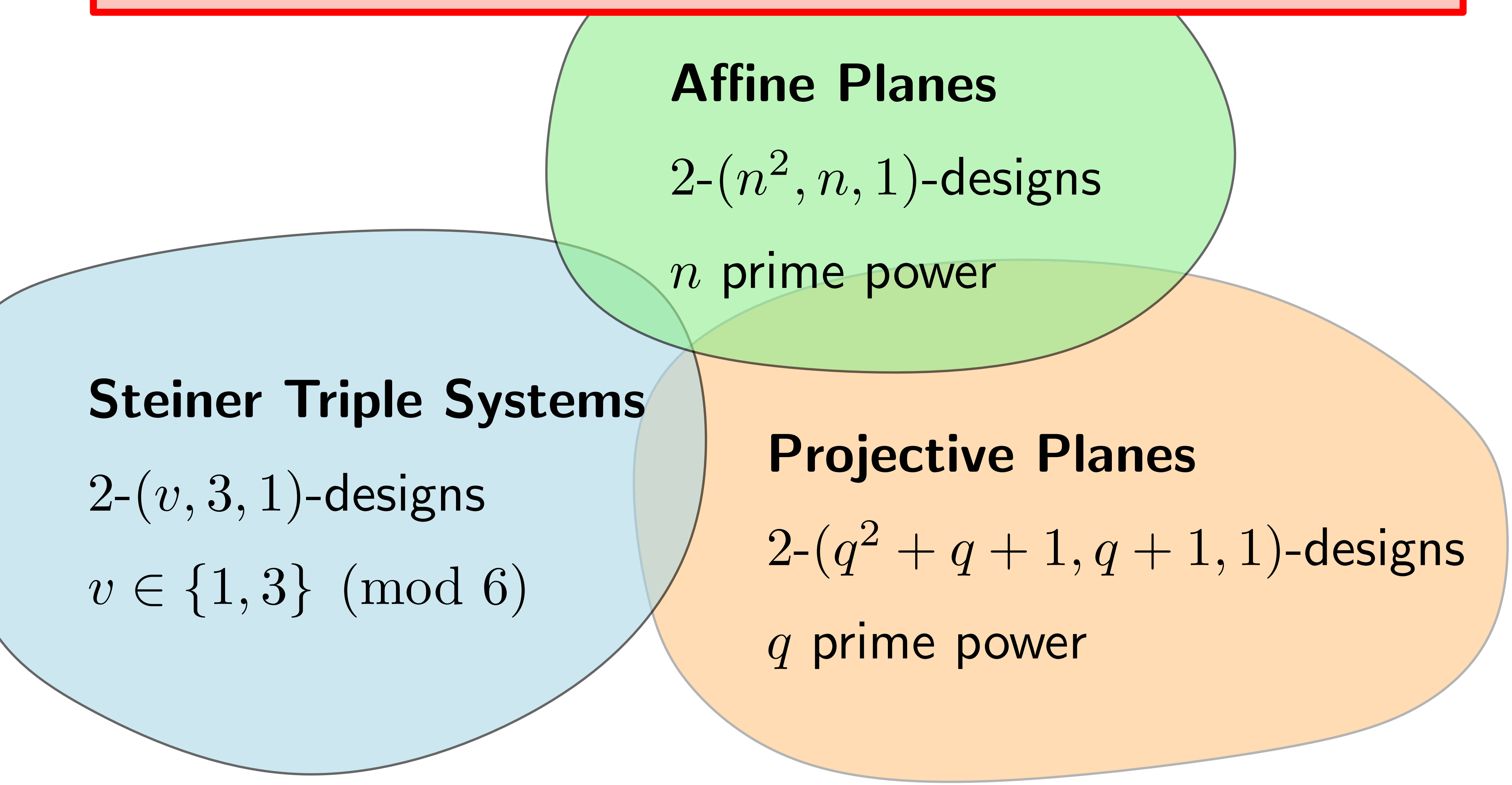
Thm: $\#$ blocks $\geq v \geq (t+1)(k-t+1)$



2-$(7, 3, 1)$-design

⑥ Designs

Existence/ Non-Existence	Steiner Triple Systems etc.	Latin Squares



⑥ Designs

Existence/ Non-Existence	Steiner Triple Systems etc.	**Latin Squares**

0	1	3	2
1	2	0	3
3	0	2	1
2	3	1	0

n-by-n array filled with $\mathbb{Z}_n$

– each row is a permutation

– each column is a permutation

Thm. $n-1$ MOLS of order n $\longleftrightarrow$ affine plane of order n^2 $\longleftrightarrow$ $n = p^k$