Problem 1. **points**

Prove that

$$\sum_{x=1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}$$

by double counting tuples $(x,(y,z))$ for integers x, y, z with $n+1 \geq x > y, z \geq 1$.

Solution.

Let N denote the number of such tuples.

First Way: If x is fixed, then there are $(x-1)$ choices from $\{1,\dots,x-1\}$ for y and z each. Hence

$$N = \sum_{x=2}^{n+1} |\{(y,z) \mid x > y, z \geq 1\}| = \sum_{x=2}^{n+1} (x-1)^2 = \sum_{x=1}^{n} x^2.$$

Second Way: We give two alternatives for counting in a second way.

Alternative 1: If (y,z) is fixed, then let $k = \max\{y,z\}$. We have $n+1-k$ choices for x from $\{k+1,\dots,n+1\}$. Note that $1 \leq y, z \leq n$. So we have

$$\begin{aligned}
N &= \sum_{(y,z)\in[n]\times[n]} |\{x \mid n+1 \geq x > y, z\}| \\
&= \sum_{(y,z)\in[n]\times[n]} |\{x \mid n+1 \geq x > \max\{y,z\}\}| \\
&= \sum_{k=1}^{n} \underbrace{|\{(y,z) \in [n]\times[n] \mid \max\{y,z\} = k\}|}_{=2k-1} \cdot \underbrace{|\{x \mid n+1 \geq x > k\}|}_{=n+1-k} \\
&= \sum_{k=1}^{n} (2k-1)(n+1-k) \\
&= \sum_{k=1}^{n} (2n+1)k + \underbrace{\sum_{k=1}^{n} (2k-1-n)}_{\overset{(1)}{=}0} - \sum_{k=1}^{n} 2k^2 \\
&= (2n+1) \underbrace{\sum_{k=1}^{n} k}_{=\frac{n(n+1)}{2}} - \sum_{k=1}^{n} 2k^2 \\
&= \frac{(2n+1)(n+1)n}{2} - 2\sum_{k=1}^{n} k^2.
\end{aligned}$$

Together with the result from the first way of counting we obtain the desired formula. To prove $\sum_{k=1}^{n}(2k-1-n) = 0$ we observe that the terms for $k=i$ and $k = n+1-i$ cancel out for $i \in [n]$, i.e.,

$$2i - 1 - n = -\left(2(n+1-i) - 1 - n\right). \tag{1}$$

Alternative 2: There are three kinds of such tuples (triples):

I: $(x,(y,z))$ with $n+1 \geq x > y > z \geq 1$,
II: $(x,(y,z))$ with $n+1 \geq x > z > y \geq 1$,
III: $(x,(y,y))$ with $n+1 \geq x > y \geq 1$.

We see that there are $\binom{n+1}{3}$ triples of kind I, $\binom{n+1}{3}$ triples of kind II, and $\binom{n+1}{2}$ triples of kind III. So

$$N = \binom{n+1}{3} + \binom{n+1}{3} + \binom{n+1}{2} = 2\frac{(n+1)n(n-1)}{6} + \frac{n(n+1)}{2} = \frac{(2n+1)(n+1)n}{6}.$$

□