Set system intersections can typically be blown up

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Abstract

We prove that given a constant \( k \geq 2 \) and a large set system \( F \) of sets of size at most \( w \), a typical \( k \)-tuple of sets \((S_1, \ldots, S_k)\) from \( F \) can be “blown up” in the following sense: for each \( 1 \leq i \leq k \), we can find a large subfamily \( F_i \) containing \( S_i \) so that for \( i \neq j \), if \( T_i \in F_i \) and \( T_j \in F_j \), then \( T_i \cap T_j = S_i \cap S_j \).

1 Introduction

1.1 Background and Main Result

Somewhat recently, the author, Lovett, Wu, and Zhang [1] improved the best known bounds for the Erdős-Rado sunflower lemma [4]. Central to this result is an “encoding” argument, in which the ground set \( X \) of a set system \( F \) is colored in some way. Most pairs \((S, C)\) of a set \( S \in F \) and a coloring \( C \) of the ground set are then shown to have some property, by the construction of an explicit injection of the “bad” pairs that do not have the property into some small collection. In this paper, we use this encoding idea to prove a rather surprising fact about set systems. A \( w \)-set system is a family of sets, each of size at most \( w \). Throughout this paper, let \( F \) be a \( w \)-set system on some ground set \( X \), and let \( k \geq 2 \) be some fixed integer.

Definition 1.1. Call a \( k \)-tuple of sets \((S_1, \ldots, S_k)\) from \( F \) \( n \)-inflatable if for each \( 1 \leq i \leq k \) there exists a subfamily \( F_i \ni S_i \) of \( F \) so that \( |F_i| \geq |F|/n \), and so that for any \( T_i \in F_i \) and \( T_j \in F_j \) with \( i \neq j \), we have \( T_i \cap T_j = S_i \cap S_j \).

We show, perhaps somewhat surprisingly, that in a large set system almost all \( k \)-tuples are inflatable, where the corresponding families \( F_i \) are very large. In the remainder of Section 1, we state the main result and several of its corollaries. The proofs of these are deferred to Section 2. In Section 3, we make some concluding remarks and pose open problems. We omit floor and ceiling signs whenever not crucial.

Theorem 1.2. Let \( k \geq 2 \) be fixed, and let \( F \) be a \( w \)-set system on some ground set \( X \). For all choices of \( n \), all but at most \( \left( \frac{k + w - 1}{k - 1} \right)^{k+\omega(k-1)} |F|^k \) of the \(|F|^k\)-tuples \((S_1, \cdots, S_k) \in F^k\) are \( n \)-inflatable.

It would be quite interesting to reproduce any of the results proved here in some other way, and perhaps this would shed some light on the sunflower conjecture.

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1.2 Proof Outline

We prove Theorem 1.2 by introducing the concept of “mimicking” sets, which allows us to capture inflatability through looking at colorings. Given a $k$-tuple $(S_1, \cdots, S_k)$ from $\mathcal{F}$ and a coloring $C$ on $X$, we say that $T$ “mimics” $S_j$ if it satisfies certain natural coloring conditions, and its intersections with the $S_i$ are the same as those of $S_j$. If enough sets mimic each $S_j$, we will have the desired inflatability property. We can use an encoding argument to count the number of “bad” $k$-tuples $(S_1, \cdots, S_k)$ so that some $S_j$ does not have too many sets $T$ mimicking it.

1.3 Blowing up Pairs of Sets

For $k = 2$, Theorem 1.2 yields the following.

Theorem 1.3 (Theorem 1.2, $k = 2$). If $\mathcal{F}$ is a $w$-set system, then all but at most $\frac{2^w(2w+2)}{n}|\mathcal{F}|^2$ pairs $(S, T)$ of sets from $\mathcal{F}$ are $n$-inflatable.

We make note of a particularly interesting consequence. First, recall the definition of a link from [1]. Given a set system $\mathcal{F}$ on $X$ and a set $U \subseteq X$, the link of $\mathcal{F}$ at $U$ is $\mathcal{F}_U = \{ S \setminus U : S \in \mathcal{F}, U \subseteq S \}$.

We are now ready to state another corollary of Theorem 1.3.

Corollary 1.4. If $S, T$ are chosen randomly from $\mathcal{F}$, then with probability at least $1 - \frac{2^w(2w+2)}{n}$, the link $\mathcal{F}_U$, where $U = S \cap T$, contains two cross-wise disjoint subfamilies $\mathcal{F}_1, \mathcal{F}_2$ each of size at least $\frac{|\mathcal{F}|}{n}$.

One could for instance take $n = 3^w$ so that the link of the intersection of almost any (as $w$ goes to infinity) pair of sets contains two cross-wise disjoint subfamilies of size $|\mathcal{F}|/3^w$. Another consequence of Corollary 1.4 is that in any induced subgraph of a Kneser graph $KG(m, w)$, a typical edge is contained in a large complete bipartite subgraph. We state this explicitly as a theorem.

Theorem 1.5. Let $G = KG(m, w)$ be a Kneser graph, and let $G'$ be an induced subgraph. Then for any $n$, and for all but at most $\frac{2^w(2w+2)}{n}|\mathcal{F}|^2$ edges $e$ of $G'$, there is some complete bipartite subgraph of $G'$ with $|V(G')|/n$ vertices on each side that contains $e$.

In particular, if we have a $w$-set system $\mathcal{F}$ with more than $\varepsilon|\mathcal{F}|^2$ pairs of disjoint sets, then $n = \frac{1}{2}2^w(2w+2)$ in Theorem 1.5 yields that there are some disjoint subfamilies $\mathcal{F}_1$ and $\mathcal{F}_2$ of size at least $\frac{\varepsilon|\mathcal{F}|}{2^{w}(2w+2)}$. In other words, any set system with many pairs of disjoint sets has a large pair of disjoint subfamilies.

1.4 Blowing up Sunflowers

We will also use Theorem 1.2 to extend the sunflower lemma of Erdős and Rado [4]. As mentioned in [1], Erdős and Rado originally called sunflowers $\Delta$-systems, but Deza and Frankl [2] coined the name “sunflower”.

Definition 1.6. For a fixed $k \geq 2$, a $k$-petal sunflower is a $k$-tuple of sets $(S_1, \cdots, S_k)$ so that for any $1 \leq i < j \leq k$, $S_i \cap S_j = S_i \cap \cdots \cap S_k$. The $S_i \setminus (S_i \cap \cdots \cap S_k)$ are called the petals, and $S_i \cap \cdots \cap S_k$ is called the kernel.

The case $k = 2$ is trivial, because any pair of sets form a 2-petal sunflower, so typically we assume $k \geq 3$.

Corollary 1.7. Let $f_k(w)$ be such that any $w$-set system with $f_k(w)$ sets contains a $k$-petal sunflower. Then, there exists some constant $D_k$ depending on $k$ so that if $\mathcal{F}$ is a $w$-set system on a ground set $X$, then almost all (as $w$ goes to infinity) of its $k$-petal sunflowers $(S_1, \cdots, S_k)$ are $n$-inflatable with parameter $n = f_k(w)^kD_k^w$.

The sunflower conjecture ([4]) states that we may take $f_k(w) = (O_k(1))^w$. By using the modifications of the original argument of the author, Lovett, Wu, and Zhang [1] due to Frankston, Kahn, Narayanan, and Park [3], Rao [5] proved that we may take $f_k(w) = (O_k(\log w))^w$. Thus, if the sunflower conjecture
is true, there is some constant $C_k$ depending on $k$ so that if $\mathcal{F}$ is a $w$-set system, then for a typical $k$-petal sunflower $(S_1, \ldots, S_k)$, there exists subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_k$ of $\mathcal{F}$ of size at least $|\mathcal{F}|/C_k^w$ so that any $S_i \in \mathcal{F}_1, \ldots, S_k \in \mathcal{F}_k$ form a sunflower. Unconditionally, there is some constant $C_k$ depending on $k$ such that a typical $k$-petal sunflower can be blown up to yield families $\mathcal{F}_1, \ldots, \mathcal{F}_k$ of $\mathcal{F}$ of size at least $|\mathcal{F}|/(C_k \log(w))^k$ so that any $S_1 \in \mathcal{F}_1, \ldots, S_k \in \mathcal{F}_k$ form a $k$-petal sunflower.

2 Proofs

We assume $w \geq 2$; $w = 1$ is a trivial case. We can assume all sets are of size exactly $w$ by adding dummy elements to sets that are too small, since this does not affect inflatability of any $k$-tuple. We may also assume the ground set $X$ has size divisible by $k$, since we may add dummy elements to $X$ to make this the case. Thus, we may randomly partition $X$ into $k$ equal-sized parts $X_i$ for $1 \leq i \leq k$. We say that the elements of $X_i$ have color $i$ and we refer to the partition of $X$ into the $X_i$ as a coloring $C$ of the ground set $X$. We call a coloring balanced if there are the same number of elements of each of the $k$ colors.

**Definition 2.1.** Fix a $w$-set system $\mathcal{F}$ on a ground set $X$, a balanced coloring $C$ of $X$, and a $k$-tuple $(S_1, \ldots, S_k)$ in $\mathcal{F}^k$. We say that $T \in \mathcal{F}$ mimics $S_j$ with respect to $(S_1, \ldots, S_k)$ if for all $i \neq j$, $S_i \cap T = S_i \cap S_j$, and $T \setminus S_j \subset X_j$.

The last condition, that all elements of $T \setminus S_j$ are colored with the color $j$, at first glance appears to be rather restrictive. It turns out that this is surprisingly not the case, in the sense that for a random $k$-tuple, many sets will mimic each $S_j$.

**Definition 2.2.** For a fixed $1 \leq j \leq k$, say that $(S_1, \ldots, S_k)$ is $n$-$j$-bad (under a balanced coloring $C$) if there are fewer than $|\mathcal{F}|/n$ sets $T \in \mathcal{F}$ such that $T$ mimics $S_j$ with respect to $(S_1, \ldots, S_k)$. Say $(S_1, \ldots, S_k)$ is $n$-bad under $C$ if it is $n$-$j$-bad under $C$ for some $1 \leq j \leq k$. A $k$-tuple is $n$-good under $C$ if it is not $n$-bad under $C$.

The following lemma shows the relationship between the good and inflatable properties:

**Lemma 2.3.** If $(S_1, \ldots, S_k)$ is $n$-good under some coloring $C$, then it is $n$-inflatable.

Therefore, it suffices to prove the following lemma:

**Lemma 2.4.** Let $\mathcal{F}$ be a set system on a ground set $X$. For any $1 \leq j \leq k$, if $C$ is a uniform balanced coloring of $X$ and $(S_1, \ldots, S_k)$ is sampled uniformly from $\mathcal{F}^k$, then $(S_1, \ldots, S_k)$ is $n$-bad with probability at most \( \binom{k+w-1}{k-1} \left( \frac{k^w}{n} \right)^k \).

The proof of Lemma 2.4 in Section 2.2 follows an encoding argument similar in spirit to that of [1]. For each fixed $1 \leq j \leq n$, this allows to explicitly bound the number of pairs of a balanced coloring $C$ and a $k$-tuple $(S_1, \ldots, S_k)$ so that $(S_1, \ldots, S_k)$ is $n$-$j$-bad under $C$.

2.1 Proof of Lemma 2.3

Take $\mathcal{F}_j$ to be the family of $T_j$ which mimic $S_j$ under the coloring $C$. Note $|\mathcal{F}_j| \geq |\mathcal{F}|/n$ for each $1 \leq j \leq k$ by assumption. It suffices to show that if $T_i \in \mathcal{F}_i$ and $T_j \in \mathcal{F}_j$ for $i \neq j$, then we have $T_i \cap T_j = S_i \cap S_j$. Clearly $T_i \supset T_i \cap S_j = S_i \cap S_j$, and similarly $T_j \supset T_j \cap S_i = S_i \cap S_j$. Hence, $T_i \cap T_j \supset S_i \cap S_j$. Now, we prove $T_i \cap T_j \subset S_i$. If $x \in T_i \cap T_j$ is not in $S_i$, then since $T_i \setminus S_i \subset X_i$, we have $x \in X_i$. Because $X_i$ and $X_j$ are disjoint, $x \notin X_j \supset T_j \setminus S_j$, and hence $x \notin T_j \setminus S_j$. Since $x \in T_j$, we have $x \in S_j$. But then $x \in T_i \cap S_j = S_i \cap S_j$, a contradiction. Hence, $T_i \cap T_j \subset S_i$, and similarly $T_i \cap T_j \subset S_j$, so $T_i \cap T_j = S_i \cap S_j$. 

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2.2 Proof of Lemma 2.4

For a fixed $1 \leq j \leq k$, we will bound the number of pairs of balanced colorings $C$ and $k$-tuples $(S_1, \cdots, S_k) \in \mathcal{F}^k$ so that $(S_1, \cdots, S_k)$ is $n$-$j$-bad under $C$. We will do this by an "encoding" argument; we can recover this pair from the following information.

1. The first piece of information will be $C'$, which is the coloring obtained by taking $C$ and then recoloring the elements of $S_j$ with the color $j$. We claim that the number of possibilities for this is at most 
\[
\binom{\mathcal{F}}{|\mathcal{F}|/k, \cdots, |\mathcal{F}|/k} (\binom{k+w-1}{k-1})^{|\mathcal{F}|/k}. \]
This is because there are at most $(k+w-1)$ possibilities for the number of times that each of the $k$ colors appears as one of the $w$ elements of $S_j$, and by the log-convexity of the factorial function, at most $(|\mathcal{F}|/k, \cdots, |\mathcal{F}|/k)$ possibilities for $C'$ once the number of elements of each color is fixed.

2. The second piece of information will be $S_i$ for each $i \neq j$. There are $|\mathcal{F}|^{k-1}$ possibilities for this.

3. The third piece of information will be $S_i \cap S_j$ for each $i \neq j$. There are at most $2^{w(k-1)}$ possibilities for this, as for each of the $k-1$ sets $S_i$ with $i \neq j$ we specify one of $2^w$ possible subsets for $S_i \cap S_j$.

4. The final piece of information specifies which of the mutually mimicking sets $T_j$ is $S_j$. The sets $T_j$ so that $T_j \cap S_i = S_j \cap S_i$ for each $i \neq j$ and so that all elements of $T_j$ have color $j$ in $C'$ must have $T_j \setminus S_j \subseteq X_j$. Hence, they must mimic $S_j$ with respect to $(S_1, \cdots, S_k)$, and so there are at most $|\mathcal{F}|/n$ possibilities for them by assumption. We can thus identify $S_j$ by a positive integer at most $|\mathcal{F}|/n$.

Thus, of the 
\[
\binom{\mathcal{F}}{|\mathcal{F}|/k, \cdots, |\mathcal{F}|/k} |\mathcal{F}|^k \text{ possible pairs of } C \text{ and } (S_1, \cdots, S_k), \]
the number so that $(S_1, \cdots, S_k)$ is $n$-$j$-bad under $C$ is at most 
\[
\binom{\mathcal{F}}{|\mathcal{F}|/k, \cdots, |\mathcal{F}|/k} |\mathcal{F}|^{k-1} (\binom{k+w-1}{k-1}) 2^{w(k-1)}|\mathcal{F}|/n \]
\[
= \left( \binom{\mathcal{F}}{|\mathcal{F}|/k, \cdots, |\mathcal{F}|/k} |\mathcal{F}|^k \right) 2^{w(k-1)} \binom{k+w-1}{w-1} \frac{1}{n}. \]

By a simple union bound over $k$ choices of $1 \leq j \leq k$, the number of possible pairs of $C$ and $(S_1, \cdots, S_k)$ so that $(S_1, \cdots, S_k)$ is $n$-bad under $C$ is at most 
\[
\left( \binom{\mathcal{F}}{|\mathcal{F}|/k, \cdots, |\mathcal{F}|/k} |\mathcal{F}|^k \right) \left( (\binom{k+w-1}{w-1}) 2^{w(k-1)} \frac{1}{n} \right). \]

This completes the proof of Lemma 2.4.

2.3 Proof of Theorem 1.2

Using Lemma 2.4, there exists a specific coloring $C$ under which at most 
\[
\binom{k+w-1}{w-1} 2^{w(k-1)} |\mathcal{F}|^k/n \]
the $|\mathcal{F}|^k$ $k$-tuples $(S_1, \cdots, S_k)$ are $n$-bad. In combination with Lemma 2.3, this immediately implies Theorem 1.2.

2.4 Proof of Corollary 1.4

If $(S,T)$ is $n$-inflatable, then there exists subfamilies $\mathcal{F}(S) \supseteq S$ and $\mathcal{F}(T) \supseteq T$ of $\mathcal{F}$ of size at least $|\mathcal{F}|/n$ such that $S' \cap T' = S \cap T = U$ for all $S' \in \mathcal{F}(S), T' \in \mathcal{F}(T)$. Then for all $S' \in \mathcal{F}(S), T' \in \mathcal{F}(T)$, we have $U \subseteq S', U \subseteq T'$, and $(S' \setminus U) \cap (T' \setminus U) = (S' \cap T') \setminus U = U \setminus U = \emptyset$. Thus if we take $\mathcal{F}_1 = \{S' \setminus U : S' \in \mathcal{F}(S)\}$ and $\mathcal{F}_2 = \{T' \setminus U : T' \in \mathcal{F}(T)\}$, we are done.
2.5 Proof of Theorem 1.5

Let $\mathcal{F} = V(G')$. Then by Theorem 1.3, all but at most $\frac{2^w(2w+2)}{n} |V(G')|^2$ pairs of sets of $\mathcal{F}$ are inflatable. If $(S,T)$ is an inflatable and $S,T$ are disjoint, then this means exactly that the edge $e$ between $S$ and $T$ in $G'$ is contained in a $K_{|V(G')|/n,|V(G')|/n}$ in $G'$.

2.6 Proof of Corollary 1.7

By Theorem 1.2, there exists a constant $C = C_k$ such that all but at most $C^w |\mathcal{F}|^k$ of the $k$-tuples of sets from $\mathcal{F}$ are $n$-inflatable. Now, if any family of $f_k(w)$ sets contains a $k$-petal sunflower, then by a simple counting argument, a $w$-set system $\mathcal{F}$ so that $|\mathcal{F}| \geq f_k(w)$ will have at least $\left( \frac{|\mathcal{F}|}{f_k(w)} \right)^k$ $k$-petal sunflowers. If $D_k = 2C_k$, then if $n = f(w)^kD_k^w$, almost all (as $w$ goes to infinity) $k$-petal sunflowers are inflatable.

3 Concluding Remarks and Open problems

The following conjecture was posed in [1].

**Conjecture 3.1.** Let $m > w \geq 1$. If two copies of the Kneser graph $KG(m,w)$ can be packed into the complete graph on $\binom{m}{w} = N$ vertices, then $w = \Omega(m)$.

In [1], it was shown that under the same assumption $w = \Omega(m/\log m)^2$.

We wonder if the property of Kneser graphs described in Theorem 1.5 is enough to prevent a packing. In particular, we propose the following generalization of Conjecture 3.1.

**Conjecture 3.2.** Let $G$ be a graph on $N$ vertices. If there exists $W$ so that for any induced subgraph $G'$ and for any $n$, all but at most $\frac{2^w}{w} |V(G')|^2$ edges of $G'$ are contained in some complete bipartite $K_{|V(G')|/n,|V(G')|/n}$, and furthermore two copies of $G$ pack into $K_N$, then $N \leq W^{O(1)}$.

Conjecture 3.1 is the special case where $W = 2^w(2w+2)$, and the $N = W^{O(1)}$ becomes $m = O(w)$.

Next, we explore consequences of Lemma 2.4. Recall the following lemma from [1], which we rephrase:

**Lemma 3.3.** ([1]) There exists a constant $C$ such that the following holds. Let $\mathcal{F}$ be a $w$-set system of size $|\mathcal{F}| \geq C^w$ over a ground set $X$. Assume that $X$ is split into $X_1$ and $X_2$ randomly.

Then with high probability, there exist distinct sets $S,T \in \mathcal{F}$ such that $S \setminus T \subset X_1$ and $T \setminus S \subset X_2$.

One can think of $X_1$ as the red elements of $X$ and $X_2$ as the blue elements of $X$; this is the terminology used in [1]. It does not matter too much if $X_1,X_2$ must be the same size or if a fair coin is flipped for each $x \in X$ to decide whether $x \in X_1$ or $x \in X_2$ as these cases can be shown to be equivalent by adding dummy elements to $X$. In any case, we can prove the following essentially equivalent lemma.

**Lemma 3.4.** There exists a constant $K$ such that the following holds. Let $\mathcal{F}$ be a $w$-set system over a ground set $X$. Assume that $X$ is split into $X_1$ and $X_2$ randomly.

Then with high probability (as $w$ goes to infinity), there exists at least $|\mathcal{F}|/K^w$ pairs of sets $S,T \in \mathcal{F}$ such that $S \setminus T \subset X_1$ and $T \setminus S \subset X_2$.

**Proof.** Let $X = X_1 \uplus X_2$ be a random partition of $X$. Call a subfamily $\mathcal{F}'$ of $\mathcal{F}$ of size $C^w$ *proper* if there are some $S,T \in \mathcal{F}'$ with $S \setminus T \subset X_1$ and $T \setminus S \subset X_2$, where $C$ is as in Lemma 3.3. By Lemma 3.3, with high probability (as $w$ goes to infinity), a random subfamily $\mathcal{F}'$ of size $C^w$ is proper, and so in a random coloring, with high probability almost all (as $w$ goes to infinity) subfamilies $\mathcal{F}'$ of size $C^w$ are proper.

But if this is the case, then in expectation a uniform random $C^w$ size subfamily $\mathcal{F}'$ contains at least $1 - o_w(1)$ pairs $(S,T)$ with $S \setminus T \subset X_1, T \setminus S \subset X_2$. By a counting argument, it follows that there are at least $|\mathcal{F}|/K^w$ such pairs of sets from $\mathcal{F}$ for some constant $K$. 

If we apply Lemma 2.4 with, say, \( n = (4K)^w \), with \( K \) from Lemma 3.4, we obtain the following:

**Corollary 3.5.** There exists some constant \( D \) so that if \( \mathcal{F} \) is a \( w \)-set system on a ground set \( X \), and \( X_1 \sqcup X_2 \) is a random partition of its ground set \( X \), then with high probability almost all (as \( w \) goes to infinity) of its pairs \( (S, T) \) with \( S \setminus T \subset X_1 \) and \( T \setminus S \subset X_2 \) are \( D^w \)-good.

In other words, if the ground set \( X \) of a large \( w \)-set system \( \mathcal{F} \) is colored red and blue, not only are there many \( S, T \) with \( S \setminus T \) all red and \( T \setminus S \) all blue, but also for almost all such pairs \( (S, T) \), \( \mathcal{F}_{S \cap T} \) contains at least \(|\mathcal{F}|/D^w \) red sets and at least \(|\mathcal{F}|/D^w \) blue sets for some absolute constant \( D \).

**Proof.** The proof is essentially the same as that of Corollary 1.4, but using Lemma 2.4 instead of Theorem 1.2 and goodness instead of inflatability. Let \( K \) be as in Lemma 3.4. Set \( n = (8K)^w \) in Lemma 2.4, so that with high probability (as \( w \) goes to infinity) all but at most \( (2K)^{-w}|\mathcal{F}|^2 \) pairs \((S, T)\) are \( n \)-good. But with high probability, at least \( K^{-w}|\mathcal{F}|^2 \) pairs \((S, T)\) have \( S \setminus T \subset X_1 \), \( T \setminus S \subset X_2 \). Set \( D = 8K \). \( \square \)

Also recall the definition of *rainbow sunflower* from [1], which we slightly generalize.

**Definition 3.6.** A *rainbow \( k \)-petal sunflower* for \( k \geq 2 \) is a \( k \)-tuple \((S_1, \ldots, S_k)\) and a partition \( X = X_1 \sqcup \cdots \sqcup X_k \) so that for all \( 1 \leq i, j \leq k \) with \( i \neq j \), \( S_i \setminus S_j \subset X_i \).

Equivalently, a rainbow sunflower is a \( k \)-petal sunflower \((S_1, \ldots, S_k)\) with kernel \( Y = S_1 \cap \cdots \cap S_k \) so that the petal \( S_i \setminus Y \) is contained in \( X_i \). The following corollary generalizes Corollary 3.5.

**Corollary 3.7.** Let \( g_k(w) \) be such that a \( w \)-set system with \( g_k(w) \) sets contains a \( k \)-petal rainbow sunflower with high probability (as \( w \) goes to infinity) under a random partition \( X = X_1 \sqcup \cdots \sqcup X_k \). Then there exists some constant \( D_k \) depending on \( k \) so that if \( \mathcal{F} \) is a \( w \)-set system on a ground set \( X \), and \( X = X_1 \sqcup \cdots \sqcup X_k \) is a random partition, then almost all (as \( w \) goes to infinity) of its rainbow sunflowers \((S_1, \ldots, S_k)\) are \( g_k(w)^k D_k^w \)-good.

**Proof.** Essentially the same as that of Corollary 1.7, but using Lemma 2.4 instead of Theorem 1.2 and goodness instead of inflatability. \( \square \)

The “rainbow sunflower conjecture” of [1] asks for the minimal \( g_k(w) \) that we may take. Note that unlike in the ordinary sunflower case, the \( k = 2 \) case is not trivial, and in fact Lemma 3.3 is exactly the statement that we may take \( g_2(w) = (O(1))^w \). It follows from [5] that we may take \( g_k(w) = (O_k(\log(w))^w \) and it is conjectured in [1] that we may take \( g_k(w) = (O_k(1))^w \).

Lastly, we propose a new conjecture aimed at solving the sunflower conjecture.

**Conjecture 3.8.** There exists a constant \( K \) such that the following holds. Let \( \mathcal{F} \geq K^w \) be a \( w \)-set system, and let \( S, S_1, S_2 \) be sampled uniformly from \( \mathcal{F} \). Then with high probability (as \( w \) goes to infinity) there exists \( T \in \mathcal{F} \) so that \( T \cap S = (S_1 \cap S) \sqcup (S_2 \cap S) \).

If we instead want to find a \( T \) with \( T \cap S \supset (S_1 \cap S) \sqcup (S_2 \cap S) \), we can do so via Theorem 1.2, although it is not actually necessary. Likewise, typically we can find a \( T \) with \( T \cap S = S_1 \cap S \), or with \( T \cap S = S_2 \cap S \).

Proving this conjecture, if it is true, may help resolve the sunflower conjecture. The idea is that we maybe can find such a set \( T_1 \) with \( T_1 \setminus S \) red, and maybe find another such set \( T_2 \) with \( T_2 \setminus S \) is blue, and then \( T_1 \) and \( T_2 \) would form a sunflower with \( S \) with \( T_1 \cap S = T_2 \cap S = (S_1 \cap S) \sqcup (S_2 \cap S) \).

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