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Note

# A short list color proof of Grötzsch's theorem

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**Abstract**

We give a short proof of the result that every planar graph of girth 5 is 3-choosable and hence also of Grötzsch's theorem saying that every planar triangle-free graph is 3-colorable.


## 1. Introduction

A classical theorem of Grötzsch [1] says that every planar triangle-free graph $G$ is 3-colorable. Moreover, if $G$ has an outer cycle of length 4 or 5, then any 3-coloring of the outer cycle can be extended to a 3-coloring of $G$. It is easy to reduce that extension to the case where $G$ has girth at least 5. (If $G$ has a separating cycle $C$ of length 4 or 5, then apply induction to its exterior and its interior. If $G$ has a facial cycle of length 4 distinct from the outer cycle, then identify two of its opposite vertices. If the outer precolored cycle has length 4, then insert a colored vertex of degree 2 on one of its edges. For details, see [4].) Thomassen [5] proves that all planar graphs of girth 5 are 3-choosable (that is, 3-list- colorable). That proof is inspired by the 5-list-color theorem in [3] but is considerably more complicated. In the present note we present a much simpler proof of a slight modification of the result in [5]. After some standard reductions on graphs of low connectivity we color a few well-chosen vertices, we delete these vertices from the graph under consideration, we delete also their colors from the color lists of their neighbors, and then the proof is completed by induction.

We apply the result and its proof of the present note in the proof of the following general 3-color theorem in [6]:

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For every fixed surface $S$, there are only finitely many 4-color-critical graphs of girth 5 on $S$. As a consequence, the chromatic number of graphs of girth 5 on $S$ can be found in polynomial time.

The notation and terminology are the same as that of [3–5].

## 2. List-coloring planar graphs of girth 5

**Theorem 2.1.** *Let $G$ be a plane graph of girth at least 5. Let $c$ be a 3-coloring of a path or cycle $P$: $v_1v_2\ldots v_q, 1\leqslant q\leqslant 6$, such that all vertices of $P$ are on the outer face boundary. For each vertex $v$ in $G$, let $L(v)$ be a list of colors. If $v$ is in $P$, then $L(v)$ consists of $c(v)$. Otherwise, $L(v)$ has at least two colors. If $v$ is not on the outer face boundary, then $L(v)$ has three colors. Assume furthermore that there is no edge joining vertices whose lists have at most two colors except for the edges in $P$. Then $c$ can be extended to an $L$-coloring of $G$, that is, a coloring such that neighbors have distinct colors, and every vertex $v$ has a color in $L(v)$.*

**Proof.** We prove Theorem 2.1 by induction on the number of vertices. We assume that $G$ is a smallest counterexample and shall reach a contradiction. We may assume that $G$ is connected, since otherwise we apply the induction hypothesis to every connected component of $G$. Similarly, $G$ has no cutvertex in $P$. Moreover, $G$ has no cutvertex at all. For, if $u$, say, is a cutvertex contained in an endblock $B$ disjoint from $P$, then we first apply the induction hypothesis to the subgraph of $G$ induced by the vertices not in $B\rightarrow u$. If $B$ has vertices with only two available colors joined to $u$, then we color each such vertex. These colored vertices of $B$ together with the edges joining them to $u$ divide $B$ into parts each of which has at most three colored vertices inducing a path. We now apply the induction hypothesis to each of those parts. This contradiction proves that $G$ is 2-connected with outer cycle $C$, say. If some edge of $P$ is a chord of $C$, then that chord divides $G$ into two parts, and we apply induction to each of those two parts. So, we may assume that no edge of $P$ is a chord of $C$, and hence we may choose the notation such that $C$: $v_1v_2\ldots v_q\ldots v_kv_1$. We claim that

$$P \text{ is a path,} \quad \text{and} \quad q+3\leqslant k. \tag{1}$$

For, if $P=C$, then we delete any vertex from $C$, and we delete the color of that vertex from the lists of its noncolored neighbors. If $C\neq P$ and $k<q+3$, then we color the vertices of $C$ not in $P$, we delete them and delete also the color of each deleted vertex from the list of each of its noncolored neighbors. We now apply the induction hypothesis to the resulting graph $G'$, if possible. As $G$ has girth at least 5, the vertices with precisely two available colors are independent. For the same reason, such a vertex cannot be joined to two vertices of $P$. But, a vertex of $G'$ with only two available colors may be joined to precisely one vertex of $P$. Then we color it (but we do not delete it). Now the colored vertices of $G'$ divide $G'$ into parts each of which has at most 6 precolored vertices and these vertices induce a path. We then apply

induction to each of those parts. This contradiction proves (1).

$C$ has no chord. (2)

For suppose (reductio ad absurdum) that $e = xy$ is a chord of $C$. Then $e$ divides $G$ into two graphs $G_1, G_2$, say. We may choose the notation such that $G_2$ has no more vertices of $P$ than $G_1$ has, and subject to that condition, $|V(G_2)|$ is minimum. We apply the induction hypothesis first to $G_1$. In particular, $x$ and $y$ receive a color. The minimality of $G_2$ implies that the outer cycle of $G_2$ is chordless. So $G_2$ has at most two vertices which have only two available colors and which are joined to one of $x, y$. We color any such vertex, and then we apply the induction hypothesis to $G_2$. This contradiction proves (2).

By a similar argument we conclude:

$G$ has no path of the form $v_j u v_i$ where $u$ is in $int(C)$, except possibly when $q = 6$ and the path is of the form $v_4 u v_7$ or $v_3 u v_k$. In particular, $u$ has only two neighbors on $C$. (3)

To prove (3) we define $G_1, G_2$ as in the proof of (2). Although $u$ may be joined to several vertices with only two available colors, the minimality of $G_2$ implies that no such vertex is in $G_2 - \{u, v_i, v_j\}$. There may be one or two vertices in $G_2 - \{u, v_i, v_j\}$ that have only two available colors and which are joined to one of $v_i, v_j$. We color any such vertex, and then at most 6 vertices of $G_2$ are colored. (If $v_j u v_i = v_3 u v_7$, and $u_8$ has only two available colors, then the path $v_1 v_2 v_3 u v_7 v_8$ is precolored in $G_2$.) If possible, we apply the induction hypothesis to $G_2$. This is possible unless the coloring of $G_1$ is not valid in $G_2$. This happens only if $v_j$ or $v_i$ has a neighbor of $P$ which is in $G_2$ but not in $G_1$. This happens only if we have one of the two exceptional cases described in (3). This contradiction proves (3).

Repeating the arguments in (2) and (3) we also get

$G$ has no path of the form $v_j u w v_i$ where $u, w$ are in $int(C)$ and $L(v_i)$ has precisely two colors. Also, $G$ has no path of the form $v_j u w v_i$ where $u, w$ are in $int(C)$, $L(v_i)$ has precisely three colors, and $j \in \{1, q\}$. (4)

Furthermore,

If $C'$ is a cycle in $G$ with at most six vertices and distinct from $C$, then the interior of $C'$ is empty. (5)

For, if (5) were false, then we apply the induction hypothesis first to $C'$ and its exterior and then to $C'$ and its interior.

If $L(v_{q+2})$ has three colors, then we complete the proof by deleting $v_q$ and deleting the color of $v_q$ from the list of each neighbor of $v_q$, and we apply the induction hypothesis to $G - v_q$ and obtain thereby a contradiction. So we assume that $L(v_{q+2})$ has at most two colors. By (1), $L(v_{q+2})$ has precisely two colors, and then $L(v_{q+3})$ has three colors. If $L(v_{q+4})$ has three colors, then we first color $v_{q+2}$ and $v_{q+1}$, then we delete them and also delete the color of $v_i$ from the list of each neighbor of $v_i$ for $i = q+2, q+1$. We obtain a contradiction by applying the induction hypothesis to the resulting graph. By (2) and (3) this is possible unless $G$ has a vertex $u$ in $int(C)$ joined to both $v_4$ and $v_7 = v_{q+1}$.

In that case we color $u$ before we apply the induction hypothesis. (Note that, if $u$ exists, then the interior of the cycle $uv_4v_5v_6v_7u$ is empty by (5).) So assume that $L(v_{q+4})$ has at most two colors. Then we give $v_{q+3}$ a color not in $L(v_{q+4})$, then we color $v_{q+2}, v_{q+1}$, and finally we delete $v_i$ and also delete the color of $v_i$ from the list of each neighbor of $v_i$ for $i = q+3, q+2, q+1$. We obtain a contradiction by applying the induction hypothesis to the resulting graph. If $q = 6$, and $G$ has a vertex $u$ in $int(C)$ joined to $v_4$ and $v_7$, then, as above, we color $u$ before we use induction. If $q = 6$, $q+3 = k$, and $G$ has a vertex $u'$ in $int(C)$ joined to $v_3$ and $v_k$, then we also color $u'$ before we use induction. Finally, there may be a path $v_{q+1}wzv_{q+3}$ where $w, z$ are $int(C)$. By (5), this path is unique. Then we color $w, z$ and delete them before we apply the induction hypothesis. Note that $u, u'$ may also exist in this case. If there are vertices joined to two colored vertices, then we also color these vertices (but do not delete them) before we apply the induction hypothesis.

The colored vertices divide $G$ into parts, and we shall argue why each part satisfies the induction hypothesis. By second statement of (4), there are at most 6 precolored vertices in each part, and they induce a path. (For, if there are vertices joined to both one of $w, z$ and a vertex $v_j$ in $P$, then $j \notin \{1, q\}$ by (4).) By (3) and the first part of (4), there is no vertex with precisely two available colors on $C$ which is joined to a vertex in $int(C)$ whose list has only two available colors after the additional coloring. As $G$ has girth at least 5, there is no other possibility for two adjacent vertices $z, z'$ to have only two available colors in their lists, as both $z, z'$ must be adjacent to a vertex that has been both colored and deleted.

This contradiction completes the proof. □ □

Matt DeVos (private comm.) has pointed out that the proof perhaps becomes slightly smoother if we require that $P$ has at most 4 vertices but then on the other hand allow a vertex with precisely two available colors to have neighbors on $P$.

D.H. Younger (private comm.) has pointed out that it would be interesting to seek a list-color version of the flow version of Grötzsch's theorem [2]

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