CONSTRUCTIVE LOWER BOUND FOR SYMMETRIC RAMSEY NUMBERS BY FRANKL AND WILSON

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Let $\mathcal{F}$ be a family of k-element subsets of an n-element set. By the result of Ray-Chaudhuri and Wilson [2],

$$\text{if } \ |\{|F \cap F'| : F, F' \in \mathcal{F}\}| \leq s, \text{ then } |\mathcal{F}| \leq \binom{n}{s}. \tag{1}$$

By the result of Frankl and Wilson [1],

$$\text{if } \ |F \cap F'| \not\equiv k \pmod{q} \text{ for a prime power } q \text{ then } |\mathcal{F}| \leq \binom{n}{q-1}. \tag{2}$$

Theorem 1 (Frankl and Wilson [1])**.** *When k is sufficiently large, $r(k) \geq \exp(\log^2 k/20 \log\log k)$.*

Proof. Let $V(G) = \binom{X}{q^2-1}$, where $|X| = q^3$ and q is a sufficiently large prime power. Let

$$E(G) = \{\{F, F'\} : |F \cap F'| \not\equiv -1 \pmod{q}\}.$$

If $F_1, \dots, F_m$ form a complete graph, then $m \leq \binom{q^3}{q-1}$ by (2). If $F_1, \dots, F_m$ form an independent set, then the pairwise intersections have sizes $q-1, 2q-1, \dots, q^2-q-1$, so $m \leq \binom{q^3}{q-1}$ by (1). So, G has no clique or co-clique on k vertices, where

$$|V(G)| = \binom{q^3}{q^2-1} \text{ and } k = \binom{q^3}{q-1}.$$

Using the bounds $\left(\frac{n}{m}\right)^m \leq \binom{n}{m} \leq n^m$, we have that

$$q^q \leq k \leq q^{3q} \text{ and } |V(G)| \geq q^{q^2/2}.$$

So $q \log q \leq \log k \leq 3q \log q$ and thus $\log k/3 \log q \leq q \leq \log k/\log q$. Therefore $\log q \leq \log\log k - \log\log q \leq \log\log k$ and thus $q \geq \log k/3 \log\log k$. Therefore

$$\begin{aligned}
|V(G)| &\geq (\log k/3 \log\log k)^{\log^2 k/18(\log\log k)^2} \\
&= \exp\left(\log^2 k \left(\log\log k - \log 3 - \log\log\log k\right)/18(\log\log k)^2\right) \\
&\geq \exp(\log^2 k/20 \log\log k).
\end{aligned}$$

□

Note that this gives that $r(k) \geq k^{c\sqrt{\log k}}$, i.e., this bound is greater than any power of k but smaller than exponential. The best constructive bound up to date is due to Barak et al., [3], $r(k) \geq \exp((1+o(1)) \log^{(2+a)k})$, for a positive constant a.

References

[1] Frankl, P. and Wilson, R. M. *Intersection theorems with geometric consequences.* Combinatorica 1 (1981), no. 4, 357–368.

[2] Ray-Chaudhuri, D. and Wilson, R. M. *On t-designs.* Osaka J. Math. 12 (1975), no. 3, 737–744.

[3] Barak, B., Rao, A., Shaltiel, R., Wigderson, A., *2-source dispersers for no(1) entropy, and Ramsey graphs beating the Frankl-Wilson construction.* Ann. of Math. (2) 176 (2012), no. 3, 1483–1543.

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