Supplementary results for Erdős-Stone theorem

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Lemma 0.1. *If $\epsilon < 1/r$ and r is a natural number, $n \geq 4/\epsilon$, $||G|| \geq (1-1/r+\epsilon)\frac{n^2}{2}$ and $|G| = n$, then there is $G' \subseteq G$ with $|G'| = n' \geq n\sqrt{r\epsilon}/2$ and $\delta(G') \geq (1-1/r+\epsilon/2)n'$.*

Proof. Delete vertices one at a time if the degree of the vertex is less than $(1-1/r+\epsilon/2)$ times the number of vertices in the remaining graph. Consider the remaining graph when x vertices are left.

Let y be the number of removed edges. Then

$$y \leq \sum_{\ell=x+1}^{n} (1-\frac{1}{r}+\frac{\epsilon}{2})\ell = (1-\frac{1}{r}+\frac{\epsilon}{2})(n+x+1)(n-x)/2 \leq (1-\frac{1}{r}+\frac{\epsilon}{2})(n^2-x^2)/2+(n-x)/2.$$

Here we use the fact that $(1-\frac{1}{r}+\frac{\epsilon}{2}) \leq 1$. Thus

$$||G|| \leq \binom{x}{2}+y \leq x^2/2+(1-\frac{1}{r}+\frac{\epsilon}{2})(n^2-x^2)/2+(n-x)/2 = (1-\frac{1}{r}+\frac{\epsilon}{2})(n^2)/2+(n-x)/2+(\frac{1}{r}-\frac{\epsilon}{2})x^2/2.$$

On the other hand, the assumptions of the Lemma give that

$$||G|| \geq (1-\frac{1}{r}+\epsilon)\frac{n^2}{2}.$$

Comparing the upper and the lower bound on $||G||$ gives:

$$\begin{aligned}
\left(1-\frac{1}{r}+\epsilon\right)\frac{n^2}{2} &\leq \left(1-\frac{1}{r}+\frac{\epsilon}{2}\right)\frac{n^2}{2}+\frac{n-x}{2}+\left(\frac{1}{r}-\frac{\epsilon}{2}\right)\frac{x^2}{2} \Longrightarrow \\
-\frac{n}{2}+\frac{\epsilon n^2}{4} &\leq \frac{-x}{2}+\left(\frac{1}{r}-\frac{\epsilon}{2}\right)\frac{x^2}{2} \Longrightarrow \\
\frac{\epsilon n^2}{8} &\leq \left(\frac{1}{r}-\frac{\epsilon}{2}\right)\frac{x^2}{2} \Longrightarrow \\
\frac{\epsilon r}{2(2-r\epsilon)}n^2 &\leq x^2 \Longrightarrow \\
\frac{\sqrt{\epsilon r}}{2}n &\leq x.
\end{aligned}$$

□

Consider W, n', n, s, t, ϵ as in David Conlon's proof of Erdős-Stone theorem. We know that

$$|W| \geq \frac{(n'-rs)(1-\frac{\epsilon}{3})s-(\frac{1}{r}-\frac{\epsilon}{2})n'sr}{(1-\frac{\epsilon}{3})s}.$$

On the other hand, we want that

$$|W| \geq \binom{s}{t}^r(t-1).$$

We also know that $n' \geq n\sqrt{r\epsilon}/2$ from the above lemma and that $s = 3t/\epsilon$.

Thus

$$\frac{(n'-rs)(1-\frac{\epsilon}{3})s-(\frac{1}{r}-\frac{\epsilon}{2})n'sr}{(1-\frac{\epsilon}{3})s}\geq\binom{s}{t}^r(t-1)$$

if

$$\frac{-r}{(1-\frac{\epsilon}{3})}+n'(1-\frac{(1-\frac{\epsilon r}{2})}{(1-\frac{\epsilon}{3})})\geq\binom{s}{t}^r(t-1)$$

if

$$n'\geq\frac{r}{(1-\frac{\epsilon}{3})}+\binom{s}{t}^r(t-1)$$

if

$$n\sqrt{r\epsilon}/2\geq 3r+2^{sr}t$$

if

$$n\sqrt{r\epsilon}/2\geq 2^{6tr/\epsilon}$$

if

$$n\geq 2^{6tr/\epsilon}\frac{1}{\epsilon}$$

if

$$n\geq 2^{7tr/\epsilon}$$

if

$$t\leq\frac{\epsilon}{7r}\log n.$$

This shows that t is of order $\log n$.

Here, we present an easy result of Bollobás and Erdős [On the structure of edge graphs, Bull. London Math. Soc, 5 (1973) 317–321] related to both extremal functions for bipartite graphs and to the Erdős-Stone theorem.

Lemma 0.2. *Let $0<\epsilon<1/2$ and $c>-2/\log(\epsilon)$. Then for every sufficiently large n there is a graph on $m=\epsilon\binom{n}{2}$ edges and n vertices that does not contain a complete bipartite graph with each part of size $c\log n$.*

Proof. For a given t, we shall count the number X of all m-vertex graphs on a specific set of n vertices that contain $K_{t,t}$. Since there are $\binom{n}{2t}\binom{2t}{t}\frac{1}{2}$ ways to choose the parts of $K_{t,t}$ in our set of n vertices,

$$X\leq\binom{n}{2t}\binom{2t}{t}\frac{1}{2}\cdot\binom{\binom{n}{2}-t^2}{m-t^2},$$

where the last term counts the number of ways to choose the remaining $m-t^2$ edges that are not in $K_{t,t}$ from the remaining $\binom{n}{2}-t^2$ available edges. If X is smaller then the total number $\binom{\binom{n}{2}}{m}$ of graphs on m edges, we see that one of these graphs does not contain $K_{t,t}$. So, we need to solve the following inequality for t:

$$\binom{n}{2t}\binom{2t}{t}\frac{1}{2}\cdot\binom{\binom{n}{2}-t^2}{m-t^2}<\binom{\binom{n}{2}}{m}.$$

This holds iff

$$\binom{n}{2t}\binom{2t}{t}\frac{1}{2}\frac{\binom{\binom{n}{2}-t^2}{m-t^2}}{\binom{\binom{n}{2}}{m}}<1.$$

Denoting $\binom{n}{2}$ by N, we see that the left hand side is bounded from above by

$$\begin{aligned}
& \frac{n^{2t}}{(2t)!}(2t)!\frac{(N-t^2)!m!}{N!(m-t^2)!} \\
\leq\ & n^{2t}\frac{1}{N}\ \frac{1}{N-1}\ \cdots\ \frac{1}{N-t^2+1}\ \frac{m}{1}\ \frac{m-1}{1}\ \cdots\ \frac{m-t^2-1}{1} \\
=\ & n^{2t}\frac{m}{N}\ \frac{m-1}{N-1}\ \cdots\ \frac{m-t^2+1}{N-t^2+1} \\
\leq\ & n^{2t}\left(\frac{m}{N}\right)^{t^2}.
\end{aligned}$$

Here, the last inequality holds since $m/N > (m-i)(N-i)$, $0 < i < t^2$. Indeed, $m/N > (m-i)(N-i)$ iff $m(N-i) > N(m-i)$ iff $nN - mi > Nm - Ni$ iff $N > m$. According to our choice, $m < \binom{n}{2} = N$.

What is left is to show that $n^{2t}\left(\frac{m}{N}\right)^{t^2} < 1$, or, equivalently that the logarithm of the expression on the left is at most 0. By plugging $t = c\log n$ and $m = \epsilon\binom{n}{2} = \epsilon N$, and using the fact that $c > -2/\log(\epsilon)$ we see that the logarithm of the expression on the left is bounded from above by

$$(2t)\log n + t^2\log\epsilon = 2c\log^2 n + c^2\log\epsilon\log^2 n \leq c\log^2 n(2 + c\log\epsilon) < c\log^2 n(2 - (2/\log\epsilon)\log\epsilon) < 0.$$

□