This writeup finishes the lecture from Friday, July 15, covering Lemma 2 from (Ramsey-) Lecture 8 from Conlon's notes. Here an extended proof is presented using the notation used in the lecture.

Let $\mathcal{H}$ be a hypergraph on n vertices, edges of size at most h (not necessarily all edges of the same size), and maximum degree d. Let $\mathcal{F}$ be a hypergraph on N vertices that is h-uniform.

- A set $S \subseteq V(\mathcal{F})$, $|S| \leq h$, is called *strong* if it is contained in more than $(1-(4d)^{|S|-h})\binom{N}{h-|S|}$ edges of $\mathcal{F}$.
- An injective map $f : V(\mathcal{H}) \to V(\mathcal{F})$ is called *strong* (wrt. $\mathcal{H}$) if $f(e)$ is strong for each edge e of $\mathcal{H}$.

Lemma 2. *If $\mathcal{F}$ has $N \geq 4n$ vertices and more than $(1-(4d)^{-h})\binom{N}{h}$ edges, then there are at least $\left(\frac{N}{2}\right)^n$ strong maps $f : V(\mathcal{H}) \to V(\mathcal{F})$ (wrt. $\mathcal{H}$).*

Proof. For a strong set $S \subseteq V(\mathcal{F})$ call a vertex $v \in V(\mathcal{F}) \setminus S$ *bad for S* if $S \cup \{v\}$ is not strong.

Claim 1. *For each strong $S \subseteq V(\mathcal{F})$, with $|S| < h$, there are at most $\frac{N}{4d}$ bad vertices in $V(\mathcal{F}) \setminus S$.*

Consider a vertex $v \in V(\mathcal{F}) \setminus S$ that is bad for S. Then $S \cup \{v\}$ is contained in at least $(4d)^{|S\cup v|-h}\binom{N}{h-|S\cup v|} = (4d)^{|S|+1-h}\binom{N}{h-|S|-1}$ <u>non</u>-edges (of size h) of $\mathcal{F}$. Moreover each non-edge (of size h) of $\mathcal{F}$ that contains S contains at most $h-|S|$ vertices that are bad for S. Let b_S denote the number of vertices in $V(\mathcal{F}) \setminus S$ that are bad for S. Then the number of non-edges (of size h) of $\mathcal{F}$ that contain S is at least

$$\frac{b_S}{h-|S|}(4d)^{|S|+1-h}\binom{N}{h-|S|-1} = \frac{b_S}{h-|S|}(4d)^{|S|+1-h}\frac{h-|S|}{N-h+|S|+1}\binom{N}{h-|S|}$$
$$= \frac{4d\, b_S}{N-h+|S|+1}(4d)^{|S|-h}\binom{N}{h-|S|}.$$

On the other hand S is strong and hence contained in at most $(4d)^{|S|-h}\binom{N}{h-|S|}$ non-edges of $\mathcal{F}$. This shows that $\frac{4d\, b_S}{N-h+|S|+1} \leq 1$ and thus proves Claim 1 as (since $|S| < h$)

$$b_S \leq \frac{N-h+|S|+1}{4d} \leq \frac{N}{4d}.$$

Fix an ordering $u_1, \ldots, u_n$ of the vertices of $\mathcal{H}$. For $i = 1, \ldots, n$ let $\mathcal{H}_i$ denote the hypergraph with vertex set $U_i = \{u_1, \ldots, u_i\}$ and edge set $\{e \cap U_i \mid e \text{ edge of } \mathcal{H}\}$ (keeping edges of size 1).

Claim 2. *For each $i = 1, \ldots, n$ there are at least $\prod_{j=0}^{i-1}(\frac{3}{4}N - j)$ strong maps $V(\mathcal{H}_i) \to V(\mathcal{F})$.*

We prove the claim by induction on i. Consider $i = 1$ for the induction basis. Note that $\emptyset \subset V(\mathcal{F})$ is strong, since $\mathcal{F}$ has more than $(1-(4d)^{-h})\binom{N}{h}$ edges. By Claim 1 there are at most $\frac{N}{4d}$ bad vertices for $\emptyset$, i.e., at least $N - \frac{N}{4d} \geq \frac{3}{4}N$ vertices are not bad for $\emptyset$. We can pick any such vertex for u_1 to obtain a strong map $f : V(\mathcal{H}_1) = \{u_1\} \to V(\mathcal{F})$. This proves the claim for $i = 1$.

Consider $i \geq 2$ and assume that the claim holds for $i-1$. Let E_i denote the set of edges in $\mathcal{H}_i$ containing u_i. Since $\mathcal{H}$ has maximum degree d, $\mathcal{H}_i$ has maximum degree at most d and hence $|E_i| \leq d$. Let $s = \prod_{j=0}^{i-2}(\frac{3}{4}N - j)$. By induction there are s strong maps $f_1, \ldots, f_s : U_{i-1} \to V(\mathcal{F})$. Consider a fixed such map $f = f_j$. For each edge $e \in E_i$ the set $e \setminus \{u_i\}$ is an edge in $\mathcal{H}_{i-1}$ and thus $f(e \setminus \{u_i\})$ is strong. By Claim 1 there are at most $\frac{N}{4d}$ bad vertices for each such set $f(e \setminus \{u_i\})$, $e \in E_i$. Since $|E_i| \leq d$, the number of vertices that are not bad for any $f(e \setminus \{u_i\})$, $e \in E_i$, is at least $N - d\frac{N}{4d} = \frac{3}{4}N$. Mapping u_i to one of these vertices (except for $f(u_1), \ldots, f(u_{i-1})$) extends f to a strong map $V(\mathcal{H}_i) \to V(\mathcal{F})$. This prove Claim 2 since the number of strong maps is at least

$$s\left(\tfrac{3}{4}N - (i-1)\right) = \prod_{j=0}^{i-1} \left(\tfrac{3}{4}N - j\right).$$

Now the proof of the lemma is complete as by Claim 2 for $i = n$ the number of strong maps $V(\mathcal{H}_n) = V(\mathcal{H}) \to F$ is at least

$$\prod_{j=0}^{n-1} \left(\tfrac{3}{4}N - j\right) = \prod_{j=0}^{\frac{1}{4}N-1} \left(\tfrac{3}{4}N - j\right) \geq \left(\tfrac{3}{4}N - \tfrac{1}{4}N\right)^{\frac{1}{4}N} = \left(\tfrac{1}{2}N\right)^{n}.$$

□