Ramsey numbers for triples

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Let $r_3(s,t)$ be the smallest integer n such that in any red/blue coloring of the triples of an n-element set, there is either an s-element set all whose triples are red or a t-element set all whose triples are blue.

Theorem 0.1. *If $s \geq 3$ and $t \geq 4$, then $r_3(s,t) \leq 2^{\binom{r(s-1,t-1)}{2}}$, $r_3(3,3) = 3$.*

Proof. Let X be an ordered set of vertices. We write $X' < X''$ for subsets X' and X'' of X if each element of X' is less than every element of X''. Let $c : \binom{X}{3} \to \{r, b\}$ be a coloring. We say that a set S of vertices makes an edge xy **happy** if $\{x,y\} < S$ and for any $z \in S$ all triples xyz are of the same color. We say that a set S of vertices makes a set X' **happy** if $X' < S$ and $c(xyz) = c(xyz')$, for any $x, y \in X'$, $z, z' \in S \cup X'$, $\{x,y\} < \{z,z'\}$.
Note that for any edge xy and any set $S' > \{x,y\}$ there is $S \subseteq S'$, such that $|S| \geq |S'|/2$ and S makes xy happy. Indeed, just take the majority color class.

Claim: For any k s.t. $n \geq 2^{\binom{k}{2}+2}$, there are sets X_k and S_k such that $|X_k| = k$, S_k makes X_k happy, and

$$|S_k| \geq \frac{n}{2^{\binom{k}{2}}} - 1.$$

Proof by induction on k. When $k = 2$, then there is such an S_2 of size at least $(n-2)/2 = n/2 - 1$. Assume that the statement is true for $k = q$, lets prove it for $k = q+1$. By induction there is a set X_q and a set S_q of elements greater than any element in X_q with $|S_q| \geq n/2^{\binom{q}{2}} - 1 \geq 3$ such that S_q makes all pairs from X_q happy and $|X_q| = q$. Let $x \in S_q$, be the smallest element and let $X_{q+1} = X_q \cup \{x\}$. We need to find a subset of S_q that makes all edges xy happy, where $y \in X_q$. Let $X_q = y_1, \ldots, y_q$. There is a subset S^1 of size $(|S_q| - 1)/2$ that makes xy_1 happy, there is a subset S^2 of S^1 that makes xy_1 and xy_2 happy and $|S^2| \geq |S^1|/2$, etc., so there is a subset S^q of size $(|S_q| - 1)/2^q$ that makes all xy happy, $y \in X_q$. Let S_{q+1} be this subset S^q. So, by induction

$$|S_{q+1}| \geq (|S_q| - 1)/2^q \geq \frac{n}{2^{\binom{q}{2}+q}} - 2 \cdot 2^{-q} \geq \frac{n}{2^{\binom{q+1}{2}}} - 1.$$

This proves the Claim.

If $s = t = 3$ then we need to force just a single red triple or a single blue triple. So, it is clear that $r_3(3,3) = 3$. Let $s \geq 3$ and $t \geq 4$. Assume that $n \geq 2^{\binom{r(s-1,t-1)}{2}}$. Let $m = r(s-1,t-1) - 1$. We see that $n \geq 2^{\binom{m+1}{2}} \geq 2^{\binom{m}{2}+2}$. By the Claim there are sets X_m and S_m, such that S_m that makes X_m happy, $|X_m| = m$ and

$$|S_m| \geq \frac{n}{2^{\binom{m}{2}}} - 1 \geq 2^{\binom{r(s-1,t-1)}{2} - \binom{r(s-1,t-1)-1}{2}} - 1 = 2^{r(s-1,t-1)-1} - 1 \geq 2.$$

Let z be the smallest element in S_m and z' be any other element of S_m. Then $\{z'\}$ makes $X_m \cup \{z\}$ happy. Note that $|X_m \cup \{z\}| = m + 1 = r(s-1, t-1)$. Let's color an edge xy in $X_m \cup \{z\}$ red if all triangles xyz' are red. Color xy blue otherwise. Then by Ramsey theorem for graphs there is a red K_{s-1} or blue K_{t-1} with vertices in $X_m \cup \{z\}$. Then $X_m \cup \{z, z'\}$, contains a set on s elements with all red triples or a set on t elements with all blue triples. □