Problem sheet 1 - Solutions

Question 1
Let S be a set of n points in $\mathbb{R}^2$ such that the distance between any pair of points is at most 1. Prove that there are at most $\lfloor \frac{n^2}{3} \rfloor$ pairs of points in S whose distance is greater than $\frac{1}{\sqrt{2}}$.

Solution Let G be the graph where the vertices correspond to the points in S and $\{u, v\} \in E(G)$ if and only if the distance between the corresponding points p_u, p_v is greater than $\frac{1}{\sqrt{2}}$.

The graph G does not contain K_4 as otherwise there are three vertices v, u, w in K_4 such that the angle between two of the points p_1, p_2 corresponding to two of the vertices from v, u, w is greater than $\frac{\pi}{2}$ (can be shown by case analysis). Then by the law of cosines we get that the distance between p_1 and p_2 is strictly greater than 1 in contradiction to the assumptions about the points. By Turán's theorem, G has at most $\lfloor \frac{n^2}{3} \rfloor$ edges and therefore there are at most $\lfloor \frac{n^2}{3} \rfloor$ pairs of points in S whose distance is greater than $\frac{1}{\sqrt{2}}$.

Question 2
Let $f(n)$ denote the largest number of edges among all triangle-free graphs on n vertices that are non-bipartite. Prove for all $n \geq 5$ that

$$f(n) = \text{ex}(n-1, K_3) + 1.$$

Solution Firstly we show that $f(n) \geq ex(n-1, K_3) + 1$. This can be done by showing an example for a graph G with $|V(G)| = n$, triangle-free and non-bipartite. Let G' be a complete bipartite graph with parts of sizes as equal as possible on $(n-1)$ vertices. Let G be a graph obtained from G' by removing an edge $\{a, b\} \in E(G)$ and adding a new vertex c connected to a and b. This graph has n vertices, $\text{ex}(n-1, K_3) + 1$ edges, it is triangle-free and it contains C_5 and hence non-bipartite.

Consider a non-bipartite, triangle-free graph G on n vertices. If $n = 5$ it is easily checked that G has at most $\text{ex}(n-1, K_3)+1 = 5$ edges. So we assume that $n \geq 6$. The graph G contains an odd cycle since it is not bipartite. Let C be a shortest odd cycle in G. By the choice of C, it is an induced cycle. Let $V' = V(G) \setminus V(C)$. Each vertex in V' has at most two neighbours in C since otherwise there is a shorter odd cycle in G. Indeed, if $v \in V'$ has three (or more) neighbours in C then one of the "segments" between two such neighbours has an even number of vertices, so yields an odd cycle together with v. Moreover $G[V']$ has no triangles and hence has at most $\mathrm{ex}(|V'|, K_3) = \lfloor \frac{|V'|^2}{4} \rfloor$ edges by Turán's theorem. Since $|V'| \leq n-5$ the number of edges in G is at most

$$\lfloor \tfrac{|V'|^2}{4} \rfloor + 2|V'| + n - |V'| \leq \lfloor \tfrac{(n-5)^2+4(n-5)+4(n-1)}{4} \rfloor + 1 = \lfloor \tfrac{1}{4}(n-1)^2 \rfloor + 1 = \mathrm{ex}(n-1, K_3) + 1,$$

as required.

Question 3

Prove that $\alpha(G) \geq \sum_v \frac{1}{d(v)+1}$. Deduce Turán's theorem.
(**Hint:** Start from considering a random ordering of the vertices of G)

Solution Let σ be a random ordering of the vertices of G, $|V(G)| = n$. For each $v \in V(G)$, let X_v be the indicator random variable which is equal to 1 if v appears before all of its neighbours in σ.

$$\mathbb{E}[X_v] = \mathbb{P}[X_v = 1] = \frac{(n-(d(v)+1))!d(v)!\binom{n}{d(v)+1}}{n!} = \frac{1}{d(v)+1}.$$

Let X be the random variable which counts the number of vertices which appear before all of their neighbours in σ. Then

$$\mathbb{E}[X] = \sum_{v \in V(G)} \mathbb{E}[X_v] = \sum_{v \in V(G)} \frac{1}{d(v)+1}.$$

The set of all vertices which appear before all of their neighbours in σ is an independent set. Hence $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d(v)+1}$. Note that

$$\sum_{v \in V(G)} \frac{1}{d(v)+1} \geq \frac{n}{\frac{\sum_{v \in V(G)} d(v)}{n} + 1}.$$

This can be shown by repeatedly applying the following inequality $\frac{1}{1+x} + \frac{1}{1+y} \geq \frac{2}{1+\frac{x+y}{2}}$.

Let G be a graph without K_{r+1}. Consider the complement graph $\overline{G}$ of G. Assume to the contrary that $|E(\overline{G})| < \binom{n}{2} - \left(1 - \frac{1}{r}\right)\frac{n^2}{2} \leq r\binom{n/r}{2}$. Then

$$\omega(G) = \alpha(\overline{G}) \geq \sum_v \frac{1}{d_{\overline{G}}(v)+1} > \frac{n}{2r\binom{n/r}{2}/n + 1} \geq r$$

which is contradiction to our assumptions.

Question 4

(a) Prove that each k-edge-connected graph contains each tree on $k+1$ vertices.

(b) Prove that each graph of girth at least $k+1$ and minimum degree d contains each tree of maximum degree d on k vertices as an induced subgraph.

Solution

(a) If a graph G is k-edge-connected, then every vertex has degree at least k. The rest of the proof as in Theorem 3 in Lecture 2 of Conlon's notes.

(b) We embed the tree greedily. Let $v_1, v_2, ..., v_k$ be the ordering of the vertices of T so each v_i, $2 \leq i \leq k$ has exactly one neighbour with a smaller index. Assume that we embedded vertices $v_1, v_2, ..., v_t$ of T and we want to embed the vertex v_{t+1}, let v_j be the unique neighbour of v_{t+1} with an index smaller than $t+1$. Let u_j be the image of v_j in G. We have that $d_G(u_j) \geq d$, hence u_j has a neighbour u which is not an image of any of the vertices $v_1, v_2, ..., v_t$. Moreover, $N(u) \cap N(u_l) = \emptyset$ for any $l \in [t]$ and an image u_l of a vertex v_l. This is case, because otherwise we obtain a cycle of length smaller than k in contradiction to the assumption on G.