Problem sheet 2

Due Monday, April 30th at 17:30.

Question 1

Consider a graph G on n vertices and m edges. Prove that G contains at least $\frac{4m}{3n}(m - \frac{n^2}{4})$ triangles.

Question 2

Let G be a graph on n vertices and let N_t, $t \geq 1$, denote the number of copies of K_t in G. Prove that

$$\frac{N_{t+1}}{N_t} \geq \frac{1}{t^2 - 1}\left(t^2 \frac{N_t}{N_{t-1}} - n\right).$$

(**Hint:** Double count pairs (A, U) such that $|A| = |U| = k$, $|A \cap U| = k - 1$, A induces K_t and U induces a non-complete graph of t vertices.)

Question 3

Let H be a graph with $\chi(H) = t$.

(a) Prove that $\mathrm{ex}(n, H) \geq \mathrm{ex}(n, K_t)$ for all $n \geq 1$.

(b) Suppose that $\mathrm{ex}(n, H) = \mathrm{ex}(n, K_t)$ for some $n \geq t$.

Prove that there is an edge e in H such that $\chi(H - e) < t$.

Question 4

Suppose that H is a graph with $\mathrm{ex}(n, H) \leq \lambda\binom{n}{2}$ for some constant λ, $0 < \lambda < 1$, and $n \geq n_0$. Prove that for any $\epsilon > 0$ and sufficiently large n any graph on n vertices and $(\lambda + \epsilon)\binom{n}{2}$ edges contains at least $c(\epsilon, n_0)n^{|V(H)|}$ copies of H.