Problem sheet 2

Due Monday, April 30th at 17:30.

Question 1

Consider a graph G on n vertices and m edges. Prove that G contains at least $\frac{4m}{3n}(m - \frac{n^2}{4})$ triangles.

Solution Let G and $|V(G)| = n, |E(G)| = m$. Let $x, y \in V(G)$, then $|N(x) \cap N(y)| \geq d(x) + d(y) - n$. Therefore, in particular, any edge $\{x, y\}$ is contained in at least $d(x) + d(y) - n$ triangles. Hence the total number of triangles in G is at least

$$\frac{1}{3} \sum_{\{x,y\} \in E(G)} (d(x) + d(y) - n) = \frac{1}{3} \sum_{x \in V(G)} d^2(x) - \frac{1}{3}nm \geq \frac{\left(\sum\limits_{x \in V(G)} d(x) \right)^2}{3n} - \frac{1}{3}nm = \frac{4m^2}{3n} - \frac{1}{3}nm.$$

Question 2

Let G be a graph on n vertices and let N_t, $t \geq 2$, denote the number of copies of K_t in G. Prove that

$$\frac{N_{t+1}}{N_t} \geq \frac{1}{t^2 - 1} \left(t^2 \frac{N_t}{N_{t-1}} - n \right).$$

(**Hint:** Double count pairs (A, U) such that $|A| = |U| = t$, $|A \cap U| = t - 1$, A induces K_t and U induces a non-complete graph of t vertices.)

Solution Let $A_1, A_2, ..., A_{N_t}$ be the collection of different K_t in G and let $B_1, B_2, ..., B_{N_{t-1}}$ be the collection of different K_{t-1} in G. Let a_i be the number of copies of K_{t+1} in G containing A_i. Let b_i be the number of copies of K_t in G containing B_i.

We double count the number N of pairs (A, U) where $A \subseteq V(G)$ induces K_t and $U \subseteq V(G)$ induces a non-complete graph on t vertices and $|A \cap U| = t - 1$. Considering, for the i^{th} copy A_i of K_t, pairs $(A_i, U_j(x))$, where $U_j(x) \subset A_i \cup \{x\}$ is a set containing a vertex $x \notin A_i$ and a vertex $y \in A_i$ that is not adjacent to x, $1 \leq j \leq t - 1$, yields

$$N \geq \sum_{i=1}^{N_t} (t - 1)(n - t - a_i).$$

This is true because there are at least $(n-t-a_i)$ vertices in G which have a non-neighbour in A_i. After choosing such a vertex v with a non-neighbour w in A_i, there are at least $(t-1)$ to remove on of the vertices in A_i which is not w.

Also we have,

$$N = \sum_{i=1}^{N_{t-1}} b_i(n-(t-1)-b_i).$$

We have $\sum_{i=1}^{N_t} a_i = (t+1)N_{t+1}$ and $\sum_{i=1}^{N_{t-1}} b_i = t\,N_t$. Using Jensen's inequality (for the concave function $f(x) = x(n-t+1-x)$) we show

$$(t-1)(n-t)N_t-(t^2-1)N_{t+1} \leq N \leq N_{t-1}\frac{\sum_{i=1}^{N_{t-1}} b_i}{N_{t-1}}\left(n-t+1-\frac{\sum_{i=1}^{N_{t-1}} b_i}{N_{t-1}}\right) = t(n-t+1)N_t - \frac{t^2N_t^2}{N_{t-1}}.$$

The above gives us the required inequality.

Question 3

Let H be a graph with $\chi(H) = t$.

(a) Prove that $\mathrm{ex}(n, H) \geq \mathrm{ex}(n, K_t)$ for all $n \geq 1$.

(b) Suppose that $\mathrm{ex}(n, H) = \mathrm{ex}(n, K_t)$ for some $n \geq t$.

Prove that there is an edge e in H such that $\chi(H-e) < t$.

Solution

(a) The complete multi-partite graph G with parts of sizes as equal as possible has chromatic number $t-1$, hence can not contain H as a subgraph. This implies that $\mathrm{ex}(n, H) \geq |E(G)|$. By Turán's theorem $|E(G)| = \mathrm{ex}(n, K_t)$.

(b) Let G be a graph as in part (a). Since $n \geq t$, the graph G is not a complete graph. Let G' be a graph obtained from G by adding an edge e' within one of the parts. Then G' contains a copy K of H, since $\mathrm{ex}(n, H) = \mathrm{ex}(n, K_t) < |E(G')|$. Since G does not contain H, the edge e' is contained in K. Let e denote the edge in H corresponding to e'. Then $H-e$ is contained in G and therefore $\chi(H-e) \leq \chi(G) = t-1 < t$.

Question 4

Suppose that H is a graph with $\mathrm{ex}(n, H) \leq \lambda\binom{n}{2}$ for some constant λ, $0 < \lambda < 1$, and $n \geq n_0$. Prove that for any $\epsilon > 0$ and sufficiently large n any graph on n vertices and $(\lambda+\epsilon)\binom{n}{2}$ edges contains at least $c(\epsilon, n_0)n^{|V(H)|}$ copies of H.

Solution Let G be a graph on n vertices and at least$(\lambda+\epsilon)\binom{n}{2}$ edges. Let t be the number of sets N on n_0 vertices and more than $\lambda\binom{n_0}{2}$ edges in $G[N]$.

We double-count pairs (e,N), where N is a set of n_0 vertices in G and e is an edge with both endpoints in N, let P be the number of such pairs. Then by first considering the edges and then choosing additional n_0-2 vertices to N we get that,

$$P=|E(G)|\binom{n-2}{n_0-2}\geq(\lambda+\epsilon)\binom{n}{2}\binom{n-2}{n_0-2}=(\lambda+\epsilon)\binom{n}{n_0}\binom{n_0}{2}.$$

On the other hand, by first considering sets of n_0 edges and then the edges in those sets we get,

$$P\leq t\binom{n_0}{2}+\left(\binom{n}{n_0}-t\right)\lambda\binom{n_0}{2}=t(1-\lambda)\binom{n_0}{2}+\lambda\binom{n}{n_0}\binom{n_0}{2}.$$

This is true because in t sets N on n_0 vertices we can have at most $\binom{n_0}{2}$ edges. On the rest $\left(\binom{n}{n_0}-t\right)$ we have at most $\lambda\binom{n_0}{2}$ edges.

These inequalities yield

$$t\geq\frac{\epsilon}{(1-\lambda)}\binom{n}{n_0}\geq\epsilon\binom{n}{n_0}.$$

For each set N on n_0 vertices and more than $\lambda\binom{n_0}{2}$ edges there is a copy of H in $G[N]$. The vertices of each copy of H in G are contained in $\binom{n-|V(H)|}{n_0-|V(H)|}$ sets on n_0 vertices. Let $v=|V(H)|$. Therefore the number of copies of H in G is at least

$$t\binom{n-v}{n_0-v}^{-1}\geq\epsilon\binom{n}{n_0}\binom{n-v}{n_0-v}^{-1}=\epsilon\,\frac{n!(n_0-v)!}{n_0!(n-v)!}\geq\epsilon\,\frac{(n_0-v)!}{n_0!}c(v)n^v.$$

For the last inequality we used that there is a constant $c(v)$, depending on v, such that $\frac{n!}{(n-v)!}\geq c(v)n^v$. This is given by Stirling's approximation, i.e., there are constants c_1 and c_2 such that for all $n>0$ we have $c_1\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\leq n!\leq c_2\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. This yields

$$\frac{n!}{(n-v)!}\geq\frac{c_1\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{c_2\sqrt{2\pi(n-v)}\left(\frac{n-v}{e}\right)^{n-v}}\geq\frac{c_1}{c_2\,e^v}\,n^v.$$